5

The Cauchy Problem and Wave Equations

"Since a general solution must be judged impossible from want of analysis, we must be content with the knowledge of some special cases, and that all the more, since the development of various cases seems to be the only way to bringing us at last to a more perfect knowledge."

Leonhard Euler

"What would geometry be without Gauss, mathematical logic without Boole, algebra without Hamilton, analysis without Cauchy?"

George Temple

5.1 The Cauchy Problem

In the theory of ordinary differential equations, by the initial-value problem we mean the problem of finding the solutions of a given differential equation with the appropriate number of initial conditions prescribed at an initial point. For example, the second-order ordinary differential equation

$$\frac{d^2u}{dt^2} = f\left(t, u, \frac{du}{dt}\right)$$

and the initial conditions

$$u\left(t_0\right) = \alpha, \qquad \left(\frac{du}{dt}\right)\left(t_0\right) = \beta,$$

constitute an initial-value problem.

An analogous problem can be defined in the case of partial differential equations. Here we shall state the problem involving second-order partial differential equations in two independent variables.

We consider a second-order partial differential equation for the function u in the independent variables x and y, and suppose that this equation can be solved explicitly for u_{yy}, and hence, can be represented in the from

$$u_{yy} = F\left(x, y, u, u_x, u_y, u_{xx}, u_{xy}\right). \tag{5.1.1}$$

For some value $y = y_0$, we prescribe the initial values of the unknown function and of the derivative with respect to y

$$u\left(x, y_0\right) = f\left(x\right), \qquad u_y\left(x, y_0\right) = g\left(x\right). \tag{5.1.2}$$

The problem of determining the solution of equation (5.1.1) satisfying the initial conditions (5.1.2) is known as the *initial-value problem.* For instance, the initial-value problem of a vibrating string is the problem of finding the solution of the wave equation

$$u_{tt} = c^2 u_{xx},$$

satisfying the initial conditions

$$u\left(x, t_0\right) = u_0\left(x\right), \qquad u_t\left(x, t_0\right) = v_0\left(x\right),$$

where $u_0\left(x\right)$ is the initial displacement and $v_0\left(x\right)$ is the initial velocity.

In initial-value problems, the initial values usually refer to the data assigned at $y = y_0$. It is not essential that these values be given along the line $y = y_0$; they may very well be prescribed along some curve L_0 in the xy plane. In such a context, the problem is called the *Cauchy problem* instead of the initial-value problem, although the two names are actually synonymous.

We consider the Euler equation

$$Au_{xx} + Bu_{xy} + Cu_{yy} = F\left(x, y, u, u_x, u_y\right), \tag{5.1.3}$$

where A, B, C are functions of x and y. Let $\left(x_0, y_0\right)$ denote points on a smooth curve L_0 in the xy plane. Also let the parametric equations of this curve L_0 be

$$x_0 = x_0\left(\lambda\right), \qquad y_0 = y_0\left(\lambda\right), \tag{5.1.4}$$

where λ is a parameter.

We suppose that two functions $f\left(\lambda\right)$ and $g\left(\lambda\right)$ are prescribed along the curve L_0. The Cauchy problem is now one of determining the solution $u\left(x, y\right)$ of equation (5.1.3) in the neighborhood of the curve L_0 satisfying the Cauchy conditions

$$u = f\left(\lambda\right), \tag{5.1.5a}$$

$$\frac{\partial u}{\partial n} = g\left(\lambda\right), \tag{5.1.5b}$$

on the curve L_0 where n is the direction of the normal to L_0 which lies to the left of L_0 in the counterclockwise direction of increasing arc length. The function $f(\lambda)$ and $g(\lambda)$ are called the *Cauchy data*.

For every point on L_0, the value of u is specified by equation (5.1.5a). Thus, the curve L_0 represented by equation (5.1.4) with the condition (5.1.5a) yields a twisted curve L in (x, y, u) space whose projection on the xy plane is the curve L_0. Thus, the solution of the Cauchy problem is a surface, called an *integral surface*, in the (x, y, u) space passing through L and satisfying the condition (5.1.5b), which represents a tangent plane to the integral surface along L.

If the function $f(\lambda)$ is differentiable, then along the curve L_0, we have

$$\frac{du}{d\lambda} = \frac{\partial u}{\partial x}\frac{dx}{d\lambda} + \frac{\partial u}{\partial y}\frac{dy}{d\lambda} = \frac{df}{d\lambda}, \tag{5.1.6}$$

and

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}\frac{dx}{dn} + \frac{\partial u}{\partial y}\frac{dy}{dn} = g, \tag{5.1.7}$$

but

$$\frac{dx}{dn} = -\frac{dy}{ds} \qquad \text{and} \qquad \frac{dy}{dn} = \frac{dx}{ds}. \tag{5.1.8}$$

Equation (5.1.7) may be written as

$$\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x}\frac{dy}{ds} + \frac{\partial u}{\partial y}\frac{dx}{ds} = g. \tag{5.1.9}$$

Since

$$\begin{vmatrix} \frac{dx}{d\lambda} & \frac{dy}{d\lambda} \\ \frac{-dy}{ds} & \frac{dx}{ds} \end{vmatrix} = \frac{(dx)^2 + (dy)^2}{ds\, d\lambda} \neq 0, \tag{5.1.10}$$

it is possible to find u_x and u_y on L_0 from the system of equations (5.1.6) and (5.1.9). Since u_x and u_y are known on L_0, we find the higher derivatives by first differentiating u_x and u_y with respect to λ. Thus, we have

$$\frac{\partial^2 u}{\partial x^2}\frac{dx}{d\lambda} + \frac{\partial^2 u}{\partial x\,\partial y}\frac{dy}{d\lambda} = \frac{d}{d\lambda}\left(\frac{\partial u}{\partial x}\right), \tag{5.1.11}$$

$$\frac{\partial^2 u}{\partial x\,\partial y}\frac{dx}{d\lambda} + \frac{\partial^2 u}{\partial y^2}\frac{dy}{d\lambda} = \frac{d}{d\lambda}\left(\frac{\partial u}{\partial y}\right). \tag{5.1.12}$$

From equation (5.1.3), we have

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} = F, \tag{5.1.13}$$

where F is known since u_x and u_y have been found. The system of equations can be solved for u_{xx}, u_{xy}, and u_{yy}, if

$$\begin{vmatrix} \frac{dx}{d\lambda} & \frac{dy}{d\lambda} & 0 \\ 0 & \frac{dx}{d\lambda} & \frac{dy}{d\lambda} \\ A & B & C \end{vmatrix} = C\left(\frac{dx}{d\lambda}\right)^2 - B\left(\frac{dx}{d\lambda}\right)\left(\frac{dy}{d\lambda}\right) + A\left(\frac{dy}{d\lambda}\right)^2 \neq 0. \quad (5.1.14)$$

The equation

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0, \quad (5.1.15)$$

is called the *characteristic equation.* It is then evident that the necessary condition for obtaining the second derivatives is that the curve L_0 must not be a characteristic curve.

If the coefficients of equation (5.1.3) and the function (5.1.5) are analytic, then all the derivatives of higher orders can be computed by the above process. The solution can then be represented in the form of a Taylor series:

$$u(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{k!\,(n-k)!}\,\frac{\partial^n u_0}{\partial x_0^k\,\partial y_0^{n-k}}\,(x-x_0)^k\,(y-y_0)^{n-k}, \quad (5.1.16)$$

which can be shown to converge in the neighborhood of the curve L_0. Thus, we may state the famous Cauchy–Kowalewskaya theorem.

5.2 The Cauchy–Kowalewskaya Theorem

Let the partial differential equation be given in the form

$$\begin{aligned} u_{yy} = F\,(&y, x_1, x_2, \ldots, x_n, u, u_y, u_{x_1}, u_{x_2} \ldots, u_{x_n}, \\ &u_{x_1 y}, u_{x_2 y}, \ldots, u_{x_n y}, u_{x_1 x_1}, u_{x_2 x_2}, \ldots, u_{x_n x_n}), \quad (5.2.1) \end{aligned}$$

and let the initial conditions

$$u = f(x_1, x_2, \ldots, x_n), \quad (5.2.2)$$

$$u_y = g(x_1, x_2, \ldots, x_n), \quad (5.2.3)$$

be given on the noncharacteristic manifold $y = y_0$.

If the function F is analytic in some neighborhood of the point $\left(y^0, x_1^0, x_2^0, \ldots, x_n^0, u^0, u_y^0, \ldots\right)$ and if the functions f and g are analytic in some neighborhood of the point $\left(x_1^0, x_2^0, \ldots, x_n^0\right)$, then the Cauchy problem has a unique analytic solution in some neighborhood of the point $\left(y^0, x_1^0, x_2^0, \ldots, x_n^0\right)$.

For the proof, see Petrovsky (1954).

The preceding statement seems equally applicable to hyperbolic, parabolic, or elliptic equations. However, we shall see that difficulties arise in formulating the Cauchy problem for nonhyperbolic equations. Consider, for instance, the famous Hadamard (1952) example.

The problem consists of the elliptic (or Laplace) equation

$$u_{xx} + u_{yy} = 0,$$

and the initial conditions on $y = 0$

$$u(x,0) = 0, \qquad u_y(x,0) = n^{-1} \sin nx.$$

The solution of this problem is

$$u(x,y) = n^{-2} \sinh ny \sin nx,$$

which can be easily verified.

It can be seen that, when n tends to infinity, the function $n^{-1} \sin nx$ tends uniformly to zero. But the solution $n^{-2} \sinh ny \sin nx$ does not become small, as n increases for any nonzero y. Physically, the solution represents an oscillation with unbounded amplitude $\left(n^{-2} \sinh ny\right)$ as $y \to \infty$ for any fixed x. Even if n is a fixed number, this solution is unstable in the sense that $u \to \infty$ as $y \to \infty$ for any fixed x for which $\sin nx \neq 0$. It is obvious then that the solution does not depend continuously on the data. Thus, it is not a properly posed problem.

In addition to existence and uniqueness, the question of continuous dependence of the solution on the initial data arises in connection with the Cauchy–Kowalewskaya theorem. It is well known that any continuous function can accurately be approximated by polynomials. We can apply the Cauchy–Kowalewskaya theorem with continuous data by using polynomial approximations only if a small variation in the initial data leads to a small change in the solution.

5.3 Homogeneous Wave Equations

To study Cauchy problems for hyperbolic partial differential equations, it is quite natural to begin investigating the simplest and yet most important equation, the one-dimensional wave equation, by the method of characteristics. The essential characteristic of the solution of the general wave equation is preserved in this simplified case.

We shall consider the following Cauchy problem of an infinite string with the initial condition

$$u_{tt} - c^2 u_{xx} = 0, \qquad x \in \mathbb{R}, \quad t > 0, \tag{5.3.1}$$

$$u(x,0) = f(x), \qquad x \in \mathbb{R}, \tag{5.3.2}$$

$$u_t(x,0) = g(x), \qquad x \in \mathbb{R}. \tag{5.3.3}$$

By the method of characteristics described in Chapter 4, the characteristic equation according to equation (4.2.4) is

$$dx^2 - c^2 dt^2 = 0,$$

which reduces to

$$dx + c\,dt = 0, \qquad dx - c\,dt = 0.$$

The integrals are the straight lines

$$x + ct = c_1, \qquad x - ct = c_2.$$

Introducing the characteristic coordinates

$$\xi = x + ct, \qquad \eta = x - ct,$$

we obtain

$$u_{xx} = u_{\xi\xi} + 2\,u_{\xi\eta} + u_{\eta\eta}, \qquad u_{tt} = c^2 \left(u_{\xi\xi} - 2\,u_{\xi\eta} + u_{\eta\eta}\right).$$

Substitution of these in equation (5.3.1) yields

$$-4c^2 u_{\xi\eta} = 0.$$

Since $c \neq 0$, we have

$$u_{\xi\eta} = 0.$$

Integrating with respect to ξ, we obtain

$$u_\eta = \psi^*(\eta),$$

where $\psi^*(\eta)$ is an arbitrary function of η. Integrating again with respect to η, we obtain

$$u(\xi,\eta) = \int \psi^*(\eta)\,d\eta + \phi(\xi).$$

If we set $\psi(\eta) = \int \psi^*(\eta)\,d\eta$, we have

$$u(\xi,\eta) = \phi(\xi) + \psi(\eta),$$

where ϕ and ψ are arbitrary functions. Transforming to the original variables x and t, we find the general solution of the wave equation

$$u(x,t) = \phi(x+ct) + \psi(x-ct), \tag{5.3.4}$$

provided ϕ and ψ are twice differentiable functions.

Now applying the initial conditions (5.3.2) and (5.3.3), we obtain

$$u(x,0) = f(x) = \phi(x) + \psi(x), \tag{5.3.5}$$

$$u_t(x,0) = g(x) = c\,\phi'(x) - c\,\psi'(x). \tag{5.3.6}$$

Integration of equation (5.3.6) gives

$$\phi(x) - \psi(x) = \frac{1}{c}\int_{x_0}^{x} g(\tau)\,d\tau + K, \tag{5.3.7}$$

where x_0 and K are arbitrary constants. Solving for ϕ and ψ from equations (5.3.5) and (5.3.7), we obtain

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(\tau)\,d\tau + \frac{K}{2},$$

$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(\tau)\,d\tau - \frac{K}{2}.$$

The solution is thus given by

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\left[\int_{x_0}^{x+ct} g(\tau)\,d\tau - \int_{x_0}^{x-ct} g(\tau)\,d\tau\right] \\ &= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\,d\tau. \end{aligned} \tag{5.3.8}$$

This is called the celebrated *d'Alembert solution* of the Cauchy problem for the one-dimensional wave equation.

It is easy to verify by direct substitution that $u(x,t)$, represented by (5.3.8), is the unique solution of the wave equation (5.3.1) provided $f(x)$ is twice continuously differentiable and $g(x)$ is continuously differentiable. This essentially proves the existence of the d'Alembert solution. By direct substitution, it can also be shown that the solution (5.3.8) is uniquely determined by the initial conditions (5.3.2) and (5.3.3). It is important to note that the solution $u(x,t)$ depends only on the initial values of f at points $x-ct$ and $x+ct$ and values of g between these two points. In other words, the solution does not depend at all on initial values outside this interval, $x-ct \le x \le x+ct$. This interval is called the *domain of dependence* of the variables (x,t).

Moreover, the solution depends continuously on the initial data, that is, the problem is well posed. In other words, a small change in either f or g results in a correspondingly small change in the solution $u(x,t)$. Mathematically, this can be stated as follows:

For every $\varepsilon > 0$ and for each time interval $0 \le t \le t_0$, there exists a number $\delta(\varepsilon, t_0)$ such that

$$|u(x,t) - u^*(x,t)| < \varepsilon,$$

whenever

$$|f(x) - f^*(x)| < \delta, \qquad |g(x) - g^*(x)| < \delta.$$

The proof follows immediately from equation (5.3.8). We have

$$\begin{aligned} |u(x,t) - u^*(x,t)| \le \frac{1}{2}\,&|f(x+ct) - f^*(x+ct)| \\ &+\frac{1}{2}\,|f(x-ct) - f^*(x-ct)| \\ &+\frac{1}{2c}\int_{x-ct}^{x+ct} |g(\tau) - g^*(\tau)|\,d\tau < \varepsilon, \end{aligned}$$

where $\varepsilon = \delta(1+t_0)$.

For any finite time interval $0 < t < t_0$, a small change in the initial data only produces a small change in the solution. This shows that the problem is well posed.

Example 5.3.1. Find the solution of the initial-value problem

$$\begin{gathered} u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = \sin x, \quad u_t(x,0) = \cos x. \end{gathered}$$

From (5.3.8), we have

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\sin(x+ct) + \sin(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \cos\tau\,d\tau \\ &= \sin x \cos ct + \frac{1}{2c}\left[\sin(x+ct) - \sin(x-ct)\right] \\ &= \sin x \cos ct + \frac{1}{c}\cos x \sin ct. \end{aligned}$$

It follows from the d'Alembert solution that, if an initial displacement or an initial velocity is located in a small neighborhood of some point (x_0, t_0), it can influence only the area $t > t_0$ bounded by two characteristics $x - ct =$ constant and $x + ct =$ constant with slope $\pm(1/c)$ passing through the point (x_0, t_0), as shown in Figure 5.3.1. This means that the initial displacement propagates with the speed $\frac{dx}{dt} = c$, whereas the effect of the initial velocity propagates at all speeds up to c. This infinite sector R in this figure is called the *range of influence* of the point (x_0, t_0).

According to (5.3.8), the value of $u(x_0, t_0)$ depends on the initial data f and g in the interval $[x_0 - ct_0,\, x_0 + ct_0]$ which is cut out of the initial line by the two characteristics $x - ct =$ constant and $x + ct =$ constant with slope $\pm(1/c)$ passing through the point (x_0, t_0). The interval $[x_0 - ct_0,\, x_0 + ct_0]$

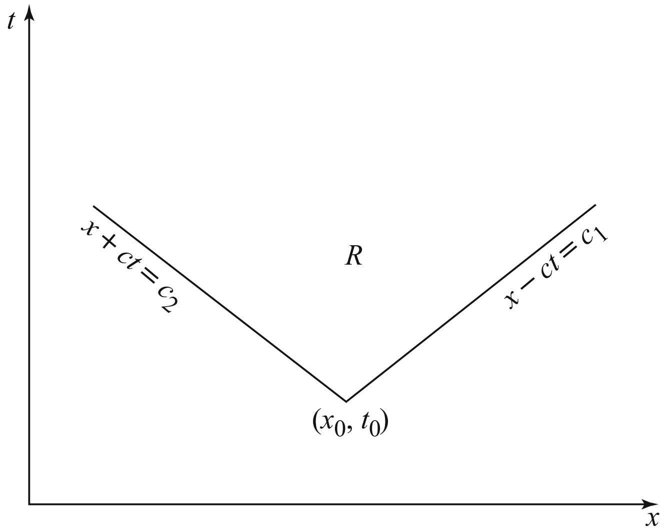

Figure 5.3.1 Range of influence

on the line $t = 0$ is called the *domain of dependence* of the solution at the point (x_0, t_0), as shown in Figure 5.3.2.

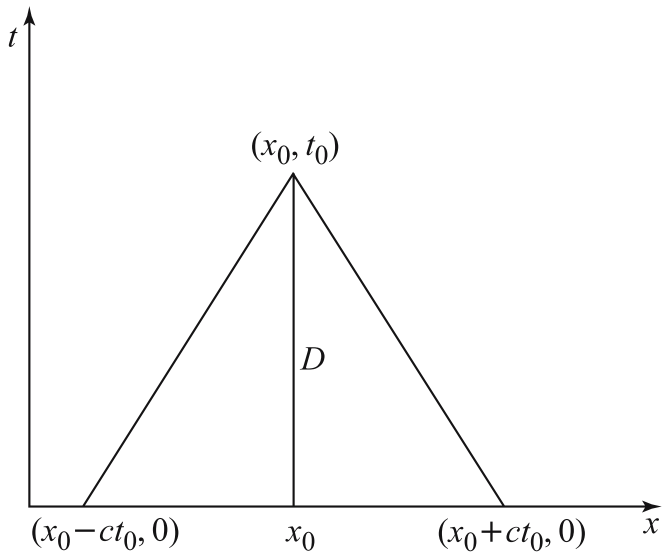

Figure 5.3.2 Domain of dependence

Since the solution $u(x,t)$ at every point (x,t) inside the triangular region D in this figure is completely determined by the Cauchy data on the interval $[x_0 - ct_0,\, x_0 + ct_0]$, the region D is called the *region of determinancy* of the solution.

We will now investigate the physical significance of the d'Alembert solution (5.3.8) in greater detail. We rewrite the solution in the form

$$u(x,t) = \frac{1}{2} f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(\tau)\,d\tau + \frac{1}{2} f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(\tau)\,d\tau. \tag{5.3.9}$$

Or, equivalently,

$$u(x,t) = \phi(x+ct) + \psi(x-ct), \tag{5.3.10}$$

where

$$\phi(\xi) = \frac{1}{2} f(\xi) + \frac{1}{2c}\int_0^{\xi} g(\tau)\,d\tau, \tag{5.3.11}$$

$$\psi(\eta) = \frac{1}{2} f(\eta) - \frac{1}{2c}\int_0^{\eta} g(\tau)\,d\tau. \tag{5.3.12}$$

Evidently, $\phi(x+ct)$ represents a progressive wave traveling in the negative x-direction with speed c without change of shape. Similarly, $\psi(x-ct)$ is also a progressive wave propagating in the positive x-direction with the same speed c without change of shape. We shall examine this point in greater detail. Treat $\psi(x-ct)$ as a function of x for a sequence of times t. At $t=0$, the shape of this function of $u=\psi(x)$. At a subsequent time, its shape is given by $u=\psi(x-ct)$ or $u=\psi(\xi)$, where $\xi = x-ct$ is the new coordinate obtained by translating the origin a distance ct to the right. Thus, the shape of the curve remains the same as time progresses, but moves to the right with velocity c as shown in Figure 5.3.3. This shows that $\psi(x-ct)$ represents a progressive wave traveling in the positive x-direction with velocity c without change of shape. Similarly, $\phi(x+ct)$ is also a progressive wave propagating in the negative x-direction with the same speed c without change of shape. For instance,

$$u(x,t) = \sin(x \mp ct) \tag{5.3.13}$$

represent sinusoidal waves traveling with speed c in the positive and negative directions respectively without change of shape. The propagation of waves without change of shape is common to all linear wave equations.

To interpret the d'Alembert formula we consider two cases:

Case 1. We first consider the case when the initial velocity is zero, that is,

$$g(x) = 0.$$

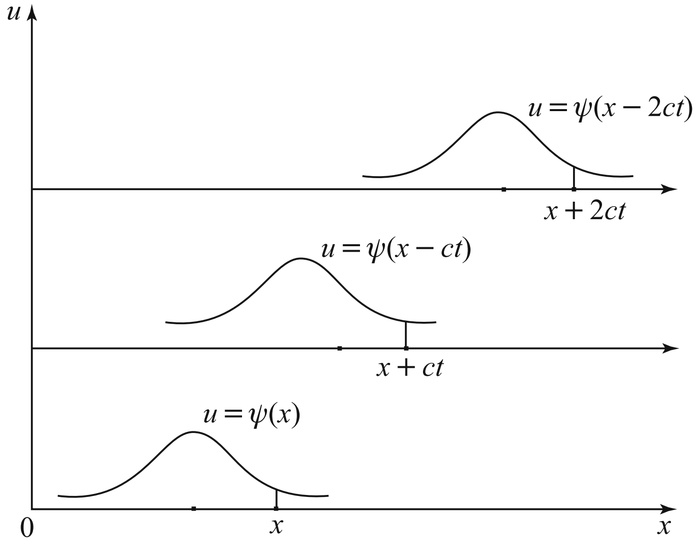

Figure 5.3.3 Progressive Waves.

Then, the d'Alembert solution has the form

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right].$$

Now suppose that the initial displacement $f(x)$ is different from zero in an interval $(-b, b)$. Then, in this case the forward and the backward waves are represented by

$$u = \frac{1}{2} f(x).$$

The waves are initially superimposed, and then they separate and travel in opposite directions.

We consider $f(x)$ which has the form of a triangle. We draw a triangle with the ordinate $x = 0$ one-half that of the given function at that point, as shown in Figure 5.3.4. If we displace these graphs and then take the sum of the ordinates of the displaced graphs, we obtain the shape of the string at any time t.

As can be seen from the figure, the waves travel in opposite directions away from each other. After both waves have passed the region of initial disturbance, the string returns to its rest position.

Case 2. We consider the case when the initial displacement is zero, that is,

$$f(x) = 0,$$

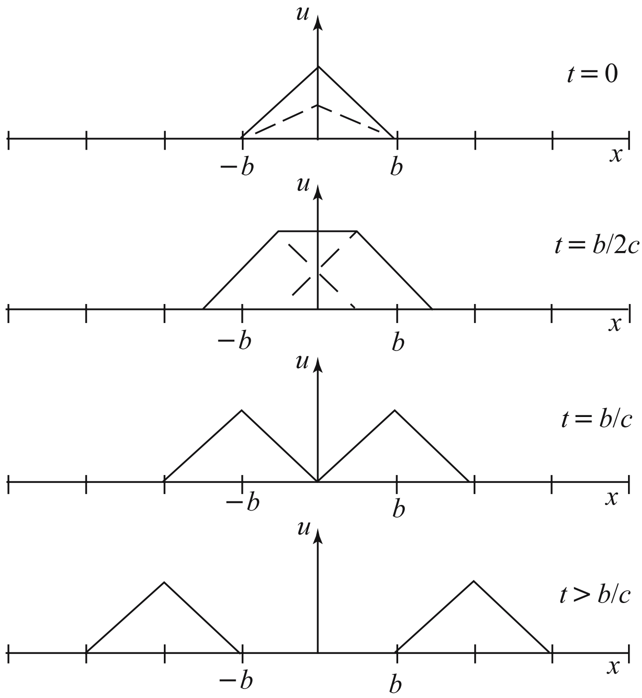

Figure 5.3.4 Triangular Waves.

and the d'Alembert solution assumes the form

$$u(x,t) = \frac{1}{2}\int_{x-ct}^{x+ct} g(\tau)\,d\tau = \frac{1}{2}\left[G(x+ct) - G(x-ct)\right],$$

where

$$G(x) = \frac{1}{c}\int_{x_0}^{x} g(\tau)\,d\tau.$$

If we take for the initial velocity

$$g(x) = \begin{cases} 0 & |x| > b \\ g_0 & |x| \le b, \end{cases}$$

then, the function $G(x)$ is equal to zero for values of x in the interval $x \le -b$, and

$$G(x) = \begin{cases} \frac{1}{c}\int_{-b}^{x} g_0\,d\tau = \frac{g_0}{c}(x+b) & \text{for } -b \le x \le b, \\ \frac{1}{c}\int_{-b}^{x} g_0\,d\tau = \frac{2bg_0}{c} & \text{for } \quad x > b. \end{cases}$$

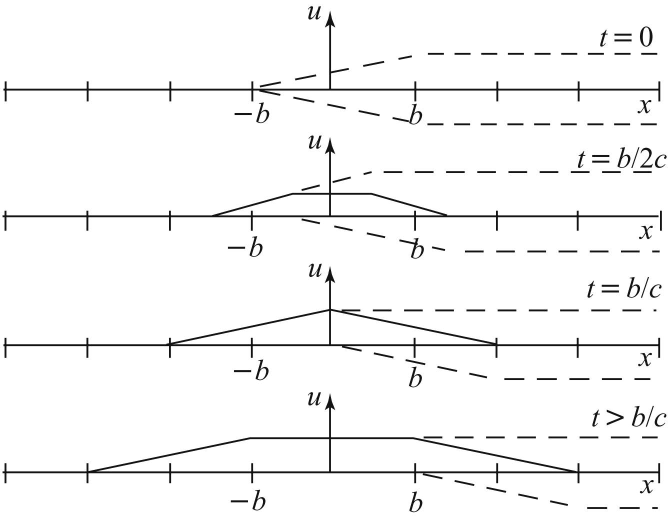

Figure 5.3.5 Graph of $u(x,t)$ at time t.

As in the previous case, the two waves which differ in sign travel in opposite directions on the x-axis. After some time t the two functions $(1/2)\,G(x)$ and $-(1/2)\,G(x)$ move a distance ct. Thus, the graph of u at time t is obtained by summing the ordinates of the displaced graphs as shown in Figure 5.3.5. As t approaches infinity, the string will reach a state of rest, but it will not, in general, assume its original position. This displacement is known as the *residual displacement*.

In the preceding examples, we note that $f(x)$ is continuous, but not continuously differentiable and $g(x)$ is discontinuous. To these initial data, there corresponds a generalized solution. By a generalized solution we mean the following:

Let us suppose that the function $u(x,t)$ satisfies the initial conditions (5.3.2) and (5.3.3). Let $u(x,t)$ be the limit of a uniformly convergent sequence of solutions $u_n(x,t)$ which satisfy the wave equation (5.3.1) and the initial conditions

$$u_n(x,0) = f_n(x), \qquad \left(\frac{\partial u_n}{\partial t}\right)(x,0) = g_n(x).$$

Let $f_n(x)$ be a continuously differentiable function, and let the sequence converge uniformly to $f(x)$; let $g_n(x)$ be a continuously differentiable function, and $\int_{x_0}^{x} g_n(\tau)\,d\tau$ approach uniformly to $\int_{x_0}^{x} g(\tau)\,d\tau$. Then, the function $u(x,t)$ is called the *generalized solution* of the problem (5.3.1)–(5.3.3).

In general, it is interesting to discuss the effect of discontinuity of the function $f(x)$ at a point $x = x_0$, assuming that $g(x)$ is a smooth function. Clearly, it follows from (5.3.8) that $u(x,t)$ will be discontinuous at each point (x, t) such that $x+ct = x_0$ or $x-ct = x_0$, that is, at each point of the two characteristic lines intersecting at the point $(x_0, 0)$. This means that discontinuities are propagated along the characteristic lines. At each point of the characteristic lines, the partial derivatives of the function $u(x, t)$ fail to exist, and hence, u can no longer be a solution of the Cauchy problem in the usual sense. However, such a function may be called a *generalized solution* of the Cauchy problem. Similarly, if $f(x)$ is continuous, but either $f'(x)$ or $f''(x)$ has a discontinuity at some point $x = x_0$, the first- or second-order partial derivatives of the solution $u(x, t)$ will be discontinuous along the characteristic lines through $(x_0, 0)$. Finally, a discontinuity in $g(x)$ at $x = x_0$ would lead to a discontinuity in the first- or second-order partial derivatives of u along the characteristic lines through $(x_0, 0)$, and a discontinuity in $g'(x)$ at x_0 will imply a discontinuity in the second-order partial derivatives of u along the characteristic lines through $(x_0, 0)$. The solution given by (5.3.8) with f, f', f'', g, and g' piecewise continuous on $-\infty < x < \infty$ is usually called the *generalized solution* of the Cauchy problem.

5.4 Initial Boundary-Value Problems

We have just determined the solution of the initial-value problem for the infinite vibrating string. We will now study the effect of a boundary on the solution.

(A) Semi-infinite String with a Fixed End

Let us first consider a semi-infinite vibrating string with a fixed end, that is,

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, && 0 < x < \infty, \quad t > 0, \\ u(x, 0) &= f(x), && 0 \leq x < \infty, \\ u_t(x, 0) &= g(x), && 0 \leq x < \infty, \\ u(0, t) &= 0, && 0 \leq t < \infty. \end{aligned} \tag{5.4.1}$$

It is evident here that the boundary condition at $x = 0$ produces a wave moving to the right with the velocity c. Thus, for $x > ct$, the solution is the same as that of the infinite string, and the displacement is influenced only by the initial data on the interval $[x - ct,\ x + ct]$, as shown in Figure 5.4.1.

When $x < ct$, the interval $[x - ct,\ x + ct]$ extends onto the negative x-axis where f and g are not prescribed.

But from the d'Alembert formula

$$u(x, t) = \phi(x + ct) + \psi(x - ct), \tag{5.4.2}$$

where

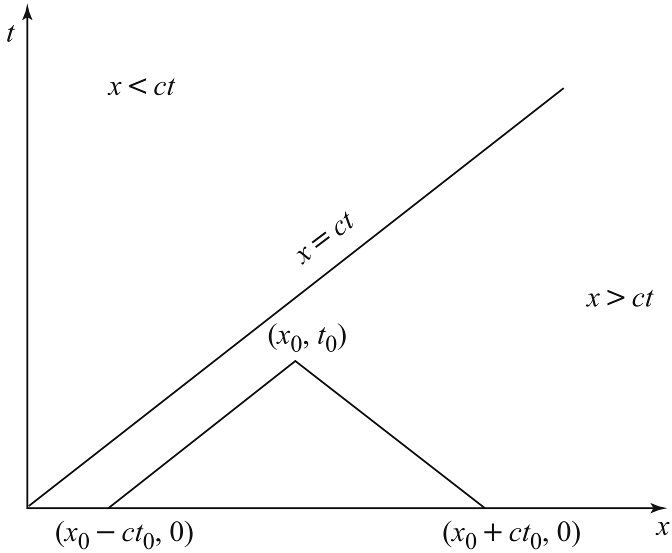

Figure 5.4.1 Displacement influenced by the initial data on $[x - ct,\ x + ct]$.

$$\phi(\xi) = \frac{1}{2} f(\xi) + \frac{1}{2c} \int_0^{\xi} g(\tau)\, d\tau + \frac{K}{2}, \tag{5.4.3}$$

$$\psi(\eta) = \frac{1}{2} f(\eta) - \frac{1}{2c} \int_0^{\eta} g(\tau)\, d\tau - \frac{K}{2}, \tag{5.4.4}$$

we see that

$$u(0, t) = \phi(ct) + \psi(-ct) = 0.$$

Hence,

$$\psi(-ct) = -\phi(ct).$$

If we let $\alpha = -ct$, then

$$\psi(\alpha) = -\phi(-\alpha).$$

Replacing α by $x - ct$, we obtain for $x < ct$,

$$\psi(x - ct) = -\phi(ct - x),$$

and hence,

$$\psi(x - ct) = -\frac{1}{2} f(ct - x) - \frac{1}{2c} \int_0^{ct - x} g(\tau)\, d\tau - \frac{K}{2}.$$

The solution of the initial boundary-value problem, therefore, is given by

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\,d\tau \quad \text{for } x > ct, \tag{5.4.5}$$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\tau)\,d\tau \quad \text{for } x < ct. \tag{5.4.6}$$

In order for this solution to exist, f must be twice continuously differentiable and g must be continuously differentiable, and in addition

$$f(0) = f''(0) = g(0) = 0.$$

Solution (5.4.6) has an interesting physical interpretation. If we draw the characteristics through the point (x_0, t_0) in the region $x > ct$, we see, as pointed out earlier, that the displacement at (x_0, t_0) is determined by the initial values on $[x_0 - ct_0, x_0 + ct_0]$.

If the point (x_0, t_0) lies in the region $x > ct$ as shown in Figure 5.4.1, we see that the characteristic $x + ct = x_0 + ct_0$ intersects the x-axis at $(x_0 + ct_0, 0)$. However, the characteristic $x - ct = x_0 - ct_0$ intersects the t-axis at $(0, t_0 - x_0/c)$, and the characteristic $x + ct = ct_0 - x_0$ intersects the x-axis at $(ct_0 - x_0, 0)$. Thus, the disturbance at $(ct_0 - x_0, 0)$ travels along the backward characteristic $x + ct = ct_0 - x_0$, and is reflected at $(0, t_0 - x_0/c)$ as a forward moving wave represented by $-\phi(ct_0 - x_0)$.

Example 5.4.1. Determine the solution of the initial boundary-value problem

$$\begin{aligned}
u_{tt} &= 4\,u_{xx}, \qquad x > 0, \qquad t > 0,\\
u(x,0) &= |\sin x|, \qquad x > 0,\\
u_t(x,0) &= 0, \qquad x \geq 0,\\
u(x,0) &= 0, \qquad t \geq 0.
\end{aligned}$$

For $x > 2t$,

$$\begin{aligned}
u(x,t) &= \frac{1}{2}\left[f(x+2t) + f(x-2t)\right]\\
&= \frac{1}{2}\left[|\sin(x+2t)| - |\sin(x-2t)|\right],
\end{aligned}$$

and for $x < 2t$,

$$\begin{aligned}
u(x,t) &= \frac{1}{2}\left[f(x+2t) - f(2t-x)\right]\\
&= \frac{1}{2}\left[|\sin(x+2t)| - |\sin(2t-x)|\right].
\end{aligned}$$

Notice that $u(0,t) = 0$ is satisfied by $u(x,t)$ for $x < 2t$ (that is, $t > 0$).

(B) Semi-infinite String with a Free End

We consider a semi-infinite string with a free end at $x = 0$. We will determine the solution of

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \qquad 0 < x < \infty, \qquad t > 0, \\ u(x,0) &= f(x), \qquad 0 \le x < \infty, \\ u_t(x,0) &= g(x), \qquad 0 \le x < \infty, \\ u_x(0,t) &= 0, \qquad 0 \le t < \infty. \end{aligned} \tag{5.4.7}$$

As in the case of the fixed end, for $x > ct$ the solution is the same as that of the infinite string. For $x < ct$, from the d'Alembert solution (5.4.2)

$$u(x,t) = \phi(x+ct) + \psi(x-ct),$$

we have

$$u_x(x,t) = \phi'(x+ct) + \psi'(x-ct).$$

Thus,

$$u_x(0,t) = \phi'(ct) + \psi'(-ct) = 0.$$

Integration yields

$$\phi(ct) - \psi(-ct) = K,$$

where K is a constant. Now, if we let $\alpha = -ct$, we obtain

$$\psi(\alpha) = \phi(-\alpha) - K.$$

Replacing α by $x - ct$, we have

$$\psi(x-ct) = \phi(ct-x) - K,$$

and hence,

$$\psi(x-ct) = \frac{1}{2} f(ct-x) + \frac{1}{2c}\int_0^{ct-x} g(\tau)\,d\tau - \frac{K}{2}.$$

The solution of the initial boundary-value problem, therefore, is given by

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\,d\tau \quad \text{for } x > ct. \tag{5.4.8}$$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(ct-x)\right] + \frac{1}{2c}\left[\int_0^{x+ct} g(\tau)\,d\tau + \int_0^{ct-x} g(\tau)\,d\tau\right] \quad \text{for } x < ct. \tag{5.4.9}$$

We note that for this solution to exist, f must be twice continuously differentiable and g must be continuously differentiable, and in addition,

$$f'(0) = g'(0) = 0.$$

Example 5.4.2. Find the solution of the initial boundary-value problem

Solve this example and at least one exercise based on this example

$$
\begin{aligned}
u_{tt} &= u_{xx}, \qquad 0 < x < \infty, \qquad t > 0, \\
u(x,0) &= \cos\left(\frac{\pi x}{2}\right), \qquad 0 \le x < \infty, \\
u_t(x,0) &= 0, \qquad 0 \le x < \infty, \\
u_x(x,0) &= 0, \qquad t \ge 0.
\end{aligned}
$$

For $x > t$

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}\left[\cos\frac{\pi}{2}(x+t) + \cos\frac{\pi}{2}(x-t)\right] \\
&= \cos\left(\frac{\pi}{2}x\right)\cos\left(\frac{\pi}{2}t\right),
\end{aligned}
$$

and for $x < t$

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}\left[\cos\frac{\pi}{2}(x+t) + \cos\frac{\pi}{2}(t-x)\right] \\
&= \cos\left(\frac{\pi}{2}x\right)\cos\left(\frac{\pi}{2}t\right).
\end{aligned}
$$

5.5 Equations with Nonhomogeneous Boundary Conditions

Nonhomogeneous Boundary condition means u(0,t) is non-zero

In the case of the initial boundary-value problems with nonhomogeneous boundary conditions, such as

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx}, \qquad x > 0, \qquad t > 0, \\
u(x,0) &= f(x), \qquad x \ge 0, \\
u_t(x,0) &= g(x), \qquad x \ge 0, \\
u(0,t) &= p(t), \qquad t \ge 0,
\end{aligned}
\tag{5.5.1}
$$

we proceed in a manner similar to the case of homogeneous boundary conditions. Using equation (5.4.2), we apply the boundary condition to obtain

$$u(0,t) = \phi(ct) + \psi(-ct) = p(t).$$

If we let $\alpha = -ct$, we have

This only for x<ct and for x>ct have the similar solution as infinite string

$$\psi(\alpha) = p\left(-\frac{\alpha}{c}\right) - \phi(-\alpha).$$

Replacing α by $x - ct$, the preceding relation becomes

$$\psi(x-ct) = p\left(t - \frac{x}{c}\right) - \phi(ct - x).$$

Thus, for $0 \leq x < ct$,

$$\begin{aligned} u(x,t) &= p\left(t - \frac{x}{c}\right) + \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\tau)\,d\tau \\ &= p\left(t - \frac{x}{c}\right) + \phi(x+ct) - \psi(ct-x), \end{aligned} \tag{5.5.2}$$

where $\phi(x + ct = \xi)$ is given by (5.3.11), and $\psi(\eta)$ is given by

$$\psi(\eta) = \frac{1}{2}f(\eta) + \frac{1}{2c}\int_0^{\eta} g(\tau)\,d\tau. \tag{5.5.3}$$

The solution for $x > ct$ is given by the solution (5.4.5) of the infinite string.

In this case, in addition to the differentiability conditions satisfied by f and g, as in the case of the problem with the homogeneous boundary conditions, p must be twice continuously differentiable in t and

$$p(0) = f(0), \qquad p'(0) = g(0), \qquad p''(0) = c^2 f''(0).$$

We next consider the initial boundary-value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, & x > 0, \quad & t > 0, \\ u(x,0) &= f(x), & x \geq 0, & \\ u_t(x,0) &= g(x), & x \geq 0, & \\ u_x(0,t) &= q(t), & t \geq 0. & \end{aligned}$$

This problem for free end nonhomogeneous condition

Using (5.4.2), we apply the boundary condition to obtain

$$u_x(0,t) = \phi'(ct) + \psi'(-ct) = q(t).$$

Then, integrating yields

$$\phi(ct) - \psi(-ct) = c\int_0^t q(\tau)\,d\tau + K.$$

If we let $\alpha = -ct$, then

$$\psi(\alpha) = \phi(-\alpha) - c\int_0^{-\alpha/c} q(\tau)\,d\tau - K.$$

Replacing α by $x - ct$, we obtain

$$\psi(x-ct) = \phi(ct-x) - c\int_0^{t-x/c} q(\tau)\,d\tau - K.$$

The solution of the initial boundary-value problem for $x < ct$, therefore, is given by

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(ct-x)\right] + \frac{1}{2c}\left[\int_0^{x+ct} g(\tau)\,d\tau + \int_0^{ct-x} g(\tau)\,d\tau\right] - c\int_0^{t-x/c} q(\tau)\,d\tau. \quad (5.5.4)$$

Here f and g must satisfy the differentiability conditions, as in the case of the problem with the homogeneous boundary conditions. In addition

$$f'(0) = q(0), \qquad g'(0) = q'(0).$$

The solution for the initial boundary-value problem involving the boundary condition

$$u_x(0,t) + h\,u(0,t) = 0, \qquad h = \text{constant}$$

can also be constructed in a similar manner from the d'Alembert solution.

5.6 Vibration of Finite String with Fixed Ends

The problem of the finite string is more complicated than that of the infinite string due to the repeated reflection of waves from the boundaries

We first consider the vibration of the string of length l fixed at both ends. The problem is that of finding the solution of

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \qquad 0 < x < l, \qquad t > 0, \\ u(x,0) &= f(x), \qquad 0 \le x \le l, \\ u_t(x,0) &= g(x), \qquad 0 \le x \le l, \\ u(0,t) &= 0, \qquad u(l,t) = 0, \qquad t \ge 0, \end{aligned} \quad (5.6.1)$$

Since both the ends are fixed therefore the position at any time is zero

From the previous results, we know that the solution of the wave equation is

$$u(x,t) = \phi(x+ct) + \psi(x-ct).$$

Applying the initial conditions, we have

$$\begin{aligned} u(x,0) &= \phi(x) + \psi(x) = f(x), \qquad 0 \le x \le l, \\ u_t(x,0) &= c\,\phi'(x) - c\,\psi'(x) = g(x), \qquad 0 \le x \le l. \end{aligned}$$

Solving for ϕ and ψ, we find

$$\phi(\xi) = \frac{1}{2}f(\xi) + \frac{1}{2c}\int_0^{\xi} g(\tau)\,d\tau + \frac{K}{2}, \qquad 0 \le \xi \le l, \quad (5.6.2)$$

$$\psi(\eta) = \frac{1}{2}f(\eta) - \frac{1}{2c}\int_0^{\eta} g(\tau)\,d\tau - \frac{K}{2}, \qquad 0 \le \eta \le l. \quad (5.6.3)$$

Hence,

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\,d\tau, \quad (5.6.4)$$

for $0 \leq x + ct \leq l$ and $0 \leq x - ct \leq l$. The solution is thus uniquely determined by the initial data in the region

$$t \leq \frac{x}{c}, \qquad t \leq \frac{l-x}{c}, \qquad t \geq 0.$$

For larger times, the solution depends on the boundary conditions. Applying the boundary conditions, we obtain

$$u(0,t) = \phi(ct) + \psi(-ct) = 0, \qquad t \geq 0, \quad (5.6.5)$$

$$u(l,t) = \phi(l+ct) + \psi(l-ct) = 0, \qquad t \geq 0. \quad (5.6.6)$$

If we set $\alpha = -ct$, equation (5.6.5) becomes

$$\psi(\alpha) = -\phi(-\alpha), \qquad \alpha \leq 0, \quad (5.6.7)$$

and if we set $\alpha = l + ct$, equation (5.6.6) takes the form

$$\phi(\alpha) = -\psi(2l - \alpha), \qquad \alpha \geq l. \quad (5.6.8)$$

With $\xi = -\eta$, we may write equation (5.6.2) as

$$\phi(-\eta) = \frac{1}{2}f(-\eta) + \frac{1}{2c}\int_0^{-\eta} g(\tau)\,d\tau + \frac{K}{2}, \qquad 0 \leq -\eta \leq l. \quad (5.6.9)$$

Thus, from (5.6.7) and (5.6.9), we have

$$\psi(\eta) = -\frac{1}{2}f(-\eta) - \frac{1}{2c}\int_0^{-\eta} g(\tau)\,d\tau - \frac{K}{2}, \qquad -l \leq \eta \leq 0. \quad (5.6.10)$$

We see that the range of $\psi(\eta)$ is extended to $-l \leq \eta \leq l$.

If we put $\alpha = \xi$ in equation (5.6.8), we obtain

$$\phi(\xi) = -\psi(2l - \xi), \qquad \xi \geq l. \quad (5.6.11)$$

Then, by putting $\eta = 2l - \xi$ in equation (5.6.3), we obtain

$$\psi(2l-\xi) = \frac{1}{2}f(2l-\xi) - \frac{1}{2c}\int_0^{2l-\xi} g(\tau)\,d\tau - \frac{K}{2}, \qquad 0 \leq 2l - \xi \leq l. \quad (5.6.12)$$

Substitution of this in equation (5.6.11) yields

$$\phi(\xi) = -\frac{1}{2} f(2l - \xi) + \frac{1}{2c} \int_0^{2l-\xi} g(\tau)\, d\tau + \frac{K}{2}, \quad l \le \xi \le 2l. \quad (5.6.13)$$

The range of $\phi(\xi)$ is thus extended to $0 \le \xi \le 2l$. Continuing in this manner, we obtain $\phi(\xi)$ for all $\xi \ge 0$ and $\psi(\eta)$ for all $\eta \le l$. Hence, the solution is determined for all $0 \le x \le l$ and $t \ge 0$.

In order to observe the effect of the boundaries on the propagation of waves, the characteristics are drawn through the end point until they meet the boundaries and then continue inward as shown in Figure 5.6.1. It can be seen from the figure that only direct waves propagate in region 1. In regions 2 and 3, both direct and reflected waves propagate. In regions, 4,5,6, ... , several waves propagate along the characteristics reflected from both of the boundaries $x = 0$ and $x = l$.

Example 5.6.1. Determine the solution of the following problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, && 0 < x < l, \quad t > 0, \\ u(x,0) &= \sin(\pi x / l), && 0 \le x \le l, \\ u_t(x,0) &= 0, && 0 \le x \le l, \\ u(0,t) &= 0, \quad u(l,t) = 0, && t \ge 0. \end{aligned}$$

Solve this and at least one exercise based on it

From equations (5.6.2) and (5.6.3), we have

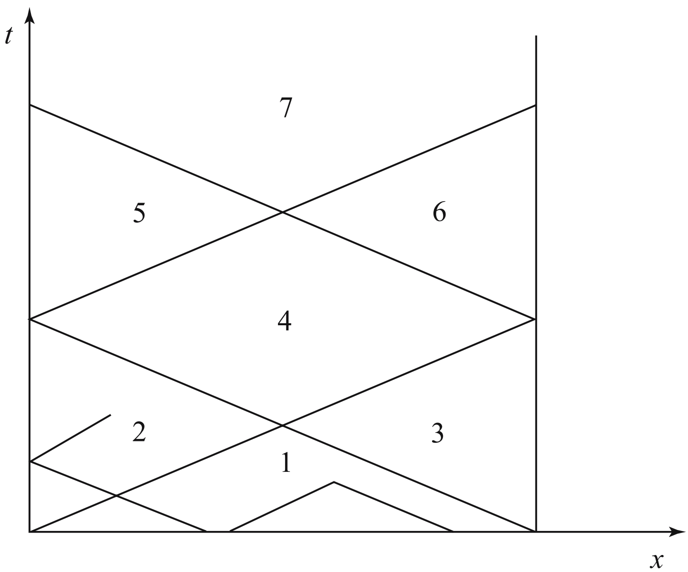

Figure 5.6.1 Regions of wave propagation.

$$\phi(\xi) = \frac{1}{2}\sin\left(\frac{\pi\xi}{l}\right) + \frac{K}{2}, \qquad 0 \le \xi \le l.$$

$$\psi(\eta) = \frac{1}{2}\sin\left(\frac{\pi\eta}{l}\right) - \frac{K}{2}, \qquad 0 \le \eta \le l.$$

Using equation (5.6.10), we obtain

$$\begin{aligned}\psi(\eta) &= -\frac{1}{2}\sin\left(-\frac{\pi\eta}{l}\right) - \frac{K}{2}, \qquad -l \le \eta \le 0\\ &= \frac{1}{2}\sin\left(\frac{\pi\eta}{l}\right) - \frac{K}{2}.\end{aligned}$$

From equation (5.6.13), we find

$$\phi(\xi) = -\frac{1}{2}\sin\left\{\frac{\pi}{l}(2l-\xi)\right\} + \frac{K}{2}, \qquad l \le \xi \le 2l.$$

Again by equation (5.6.7) and from the preceding $\phi(\xi)$, we have

$$\phi(\eta) = \frac{1}{2}\sin\left(\frac{\pi\eta}{l}\right) - \frac{K}{2}, \qquad -2l \le \eta \le -l.$$

Proceeding in this manner, we determine the solution

$$\begin{aligned}u(x,t) &= \phi(\xi) + \psi(\eta)\\ &= \frac{1}{2}\left[\sin\frac{\pi}{l}(x+ct) + \sin\frac{\pi}{l}(x-ct)\right]\end{aligned}$$

for all x in $(0,l)$ and for all $t > 0$.

Similarly, the solution of the finite initial boundary-value problem

$$\begin{aligned}u_{tt} &= c^2 u_{xx}, \qquad 0 < x < l, \qquad t > 0,\\ u(x,0) &= f(x), \qquad 0 \le x \le l,\\ u_t(x,0) &= g(x), \qquad 0 \le x \le l,\\ u(0,t) &= p(t), \qquad u(l,t) = q(t), \qquad t \ge 0,\end{aligned}$$

can be determined by the same method.

5.7 Nonhomogeneous Wave Equations

We shall consider next the Cauchy problem for the nonhomogeneous wave equation

$$u_{tt} = c^2 u_{xx} + h^*(x,t), \tag{5.7.1}$$

with the initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = g^*(x). \tag{5.7.2}$$

Using the transformation y=ct transform u_tt=u_yy(y_t)^2+u_y y_tt=c^2u_yy+0 put it into (5.7.1) we got (5.7.4)

u_t=u_y y_t=cu_y

By the coordinate transformation

Boundary condition (5.7.2) converted into (5.7.6)

$$y = ct, \tag{5.7.3}$$

the problem is reduced to

C.E. of (5.7.4) dy/dx=(B(+-)(B^2-4AC)^1/2)/2A=(0(+-)(0-4.1.(-1))^1/2)/2=+-1

$$u_{xx} - u_{yy} = h(x, y), \tag{5.7.4}$$

$$u(x, 0) = f(x), \tag{5.7.5}$$

$$u_y(x, 0) = g(x), \tag{5.7.6}$$

where $h(x, y) = -h^*/c^2$ and $g(x) = g^*/c$.

C.E.'s are dy/dx=1 and dy/dx=-1 this implies x+y=c_1 & x-y=c_2

Let $P_0(x_0, y_0)$ be a point of the plane, and let Q_0 be the point $(x_0, 0)$ on the initial line $y = 0$. Then the characteristics, $x \pm y =$ constant, of equation (5.7.4) are two straight lines drawn through the point P_0 with slopes ± 1. Obviously, they intersect the x-axis at the points $P_1(x_0 - y_0, 0)$ and $P_2(x_0 + y_0, 0)$, as shown in Figure 5.7.1. Let the sides of the triangle $P_0P_1P_2$ be designated by B_0, B_1, and B_2, and let D be the region representing the interior of the triangle and its boundaries B. Integrating both sides of equation (5.7.4), we obtain

$$\iint_R (u_{xx} - u_{yy})\, dR = \iint_R h(x, y)\, dR. \tag{5.7.7}$$

Now we apply Green's theorem to obtain

Line B1 is x+y=c1 with slop -1 and B2 is x-y=c2 with slop +1; Let intersection of B1 & B2 is P0 at (x0,y0)

P0P1=P1Q0 & P0P2=P2Q0

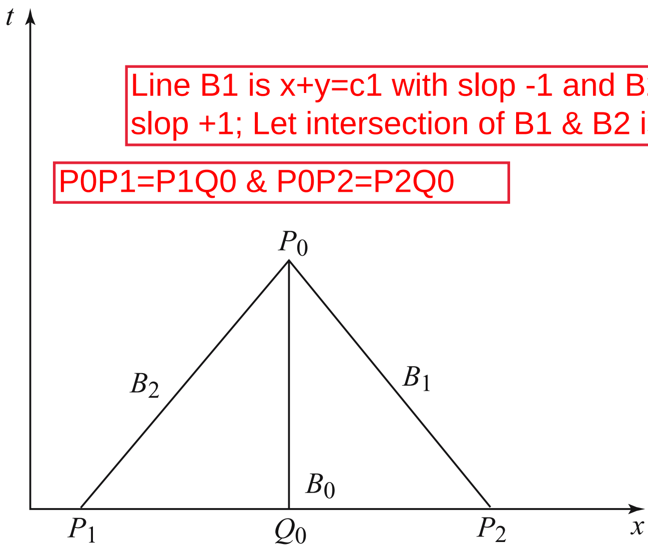

Figure 5.7.1 Triangular Region.

$$\iint_R (u_{xx} - u_{yy})\, dR = \oint_B (u_x dy + u_y dx)\,. \tag{5.7.8}$$

Since B is composed of B_0, B_1, and B_2, we note that

$$\begin{aligned}
\int_{B_0} (u_x\, dy + u_y\, dx) &= \int_{x_0-y_0}^{x_0+y_0} u_y\, dx, \\
\int_{B_1} (u_x\, dy + u_y\, dx) &= \int_{B_1} (-u_x\, dx - u_y\, dy)\,, \\
&= u\,(x_0 + y_0, 0) - u\,(x_0, y_0)\,, \\
\int_{B_2} (u_x\, dy + u_y\, dx) &= \int_{B_2} (u_x\, dx + u_y\, dy)\,, \\
&= u\,(x_0 - y_0, 0) - u\,(x_0, y_0)\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
\oint_B (u_x\, dy + u_y\, dx) = -2\, u\,(x_0, y_0) + u\,(x_0 - y_0, 0) \\
+ u\,(x_0 + y_0, 0) + \int_{x_0-y_0}^{x_0+y_0} u_y\, dx.
\end{aligned} \tag{5.7.9}$$

Combining equations (5.7.7), (5.7.8) and (5.7.9), we obtain

$$\begin{aligned}
u\,(x_0, y_0) = \frac{1}{2}\,[u\,(x_0 + y_0, 0) + u\,(x_0 - y_0, 0)] \\
+ \frac{1}{2}\int_{x_0-y_0}^{x_0+y_0} u_y\, dx - \frac{1}{2}\iint_R h\,(x, y)\, dR.
\end{aligned} \tag{5.7.10}$$

We have chosen x_0, y_0 arbitrarily, and as a consequence, we replace x_0 by x and y_0 by y. Equation (5.7.10) thus becomes

$$u\,(x, y) = \frac{1}{2}\,[f\,(x + y) + f\,(x - y)] + \frac{1}{2}\int_{x-y}^{x+y} g\,(\tau)\, d\tau - \frac{1}{2}\iint_R h\,(x, y)\, dR.$$

In terms of the original variables

$$u\,(x, t) = \frac{1}{2}\,[f\,(x + ct) + f\,(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g^*\,(\tau)\, d\tau - \frac{1}{2}\iint_R h\,(x, t)\, dR. \tag{5.7.11}$$

Example 5.7.1. Determine the solution of

Solve it and at least one exercise based on it

$$\begin{aligned}
u_{xx} - u_{yy} &= 1, \\
u\,(x, 0) &= \sin x, \\
u_y\,(x, 0) &= x.
\end{aligned}$$

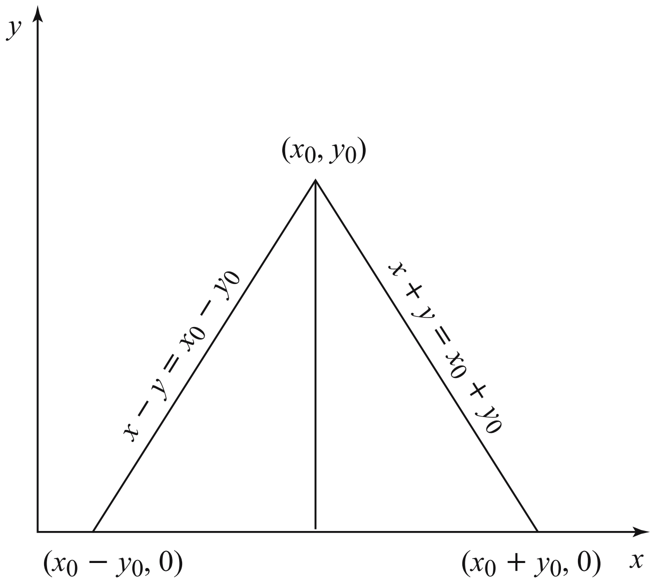

Figure 5.7.2 Triangular Region.

It is easy to see that the characteristics are $x+y = \text{constant} = x_0 + y_0$ and $x - y = \text{constant} = x_0 - y_0$, as shown in Figure 5.7.2. Thus,

$$\begin{aligned} u(x_0, y_0) &= \frac{1}{2}\left[\sin(x_0+y_0) + \sin(x_0-y_0)\right] \\ &\quad + \frac{1}{2}\int_{x_0-y_0}^{x_0+y_0} \tau\, d\tau - \frac{1}{2}\int_0^{y_0}\int_{y+x_0-y_0}^{-y+x_0+y_0} dx\, dy \\ &= \frac{1}{2}\left[\sin(x_0+y_0) + \sin(x_0-y_0)\right] + x_0 y_0 - \frac{1}{2}\, y_0^2. \end{aligned}$$

Now dropping the subscript zero, we obtain the solution

$$u(x, y) = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right] + xy - \frac{1}{2}\, y^2.$$

5.8 The Riemann Method

We shall discuss Riemann's method of integrating the linear hyperbolic equation

$$L[u] \equiv u_{xy} + a u_x + b u_y + c u = f(x, y), \tag{5.8.1}$$

where L denotes the linear operator, and $a(x,y)$, $b(x,y)$, $c(x,y)$, and $f(x,y)$ are differentiable functions in some domain D^*. The method consists essentially of the derivation of an integral formula which represents the solution of the Cauchy problem.

Let $v(x,y)$ be a function having continuous second-order partial derivatives. Then, we may write

$$\begin{aligned} vu_{xy} - uv_{xy} &= (vu_x)_y - (vu_y)_x\,, \\ vau_x &= (avu)_x - u\,(av)_x\,, \\ vbu_y &= (bvu)_y - u\,(bv)_y\,, \end{aligned} \tag{5.8.2}$$

so that

1st equation could be VU_xy-UV_xy=(VU_x)_y-(UV_y)_x

$$vL[u] - uM[v] = U_x + V_y, \tag{5.8.3}$$

where M is the operator represented by

$$M[v] = v_{xy} - (av)_x - (bv)_y + cv, \tag{5.8.4}$$

and

$$U = auv - uv_y, \qquad V = buv + vu_x. \tag{5.8.5}$$

The operator M is called the *adjoint operator* of L. If $M = L$, then the operator L is said to be *self-adjoint*. Now applying Green's theorem, we have

$$\iint_D (U_x + V_y)\, dx\, dy = \oint_C (U\, dy - V\, dx)\,, \tag{5.8.6}$$

where C is the closed curve bounding the region of integration D which is in D^*.

Let Λ be a smooth initial curve which is continuous, as shown in Figure 5.8.1. Since equation (5.8.1) is in first canonical form, x and y are the characteristic coordinates. We assume that the tangent to Λ is nowhere parallel to the x or y axis. Let $P(\alpha,\beta)$ be a point at which the solution to the Cauchy problem is sought. Line PQ parallel to the x axis intersects the initial curve Λ at Q, and line PR parallel to the y axis intersects the curve Λ at R. We suppose that u and u_x or u_y are prescribed along Λ.

Let C be the closed contour $PQRP$ bounding D. Since $dy = 0$ on PQ and $dx = 0$ on PR, it follows immediately from equations (5.8.3) and (5.8.6) that

$$\iint_D (vL[u] - uM[v])\, dx\, dy = \int_Q^R (U\, dy - V\, dx) + \int_R^P U\, dy - \int_P^Q V\, dx. \tag{5.8.7}$$

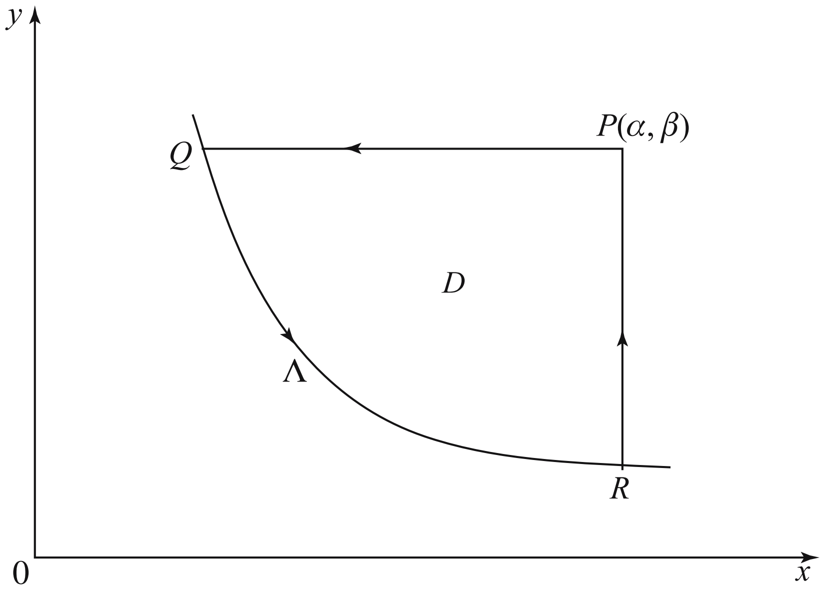

Figure 5.8.1 Smooth initial curve.

From equation (5.8.5), we find

$$\int_P^Q V\,dx = \int_P^Q bvu\,dx + \int_P^Q vu_x\,dx.$$

Integrating by parts, we obtain

$$\int_P^Q vu_x dx = [uv]_P^Q - \int_P^Q uv_x dx.$$

Hence, we may write

$$\int_P^Q V\,dx = [uv]_P^Q + \int_P^Q u\,(bv - v_x)\,dx.$$

Substitution of this integral in equation (5.8.7) yields

$$[uv]_P = [uv]_Q + \int_P^Q u\,(bv - v_x)\,dx - \int_R^P u\,(av - v_y)\,dy - \int_Q^R (U\,dy - V\,dx) + \iint_D (vL\,[u] - uM\,[v])\,dx\,dy. \tag{5.8.8}$$

Suppose we can choose the function $v\,(x, y; \alpha, \beta)$ to be the solution of the adjoint equation

$$M[v] = 0, \tag{5.8.9}$$

satisfying the conditions

$$\begin{aligned} v_x &= bv \quad \text{when} \quad y = \beta, \\ v_y &= av \quad \text{when} \quad x = \alpha, \\ v &= 1 \quad \text{when} \quad x = \alpha \quad \text{and} \quad y = \beta. \end{aligned} \tag{5.8.10}$$

The function $v(x, y; \alpha, \beta)$ is called the *Riemann function*. Since $L[u] = f$, equation (5.8.8) reduces to,

$$[u]_P = [uv]_Q - \int_Q^R uv\,(a\,dy - b\,dx) + \int_Q^R (uv_y dy + vu_x dx) + \iint_D vf\,dx\,dy. \tag{5.8.11}$$

This gives us the value of u at the point P when u and u_x are prescribed along the curve Λ. When u and u_y are prescribed, the identity

$$[uv]_R - [uv]_Q = \int_Q^R \left\{(uv)_x\,dx + (uv)_y\,dy\right\},$$

may be used to put equation (5.8.8) in the form

$$[u]_P = [uv]_R - \int_Q^R uv\,(a\,dy - b\,dx) - \int_Q^R (uv_x dx + vu_y dy) + \iint_D vf\,dx\,dy. \tag{5.8.12}$$

By adding equations (5.8.11) and (5.8.12), the value of u at P is given by

$$[u]_P = \frac{1}{2}\left([uv]_Q + [uv]_R\right) - \int_Q^R uv\,(a\,dy - b\,dx) - \frac{1}{2}\int_Q^R u\,(v_x dx - v_y dy) + \frac{1}{2}\int_Q^R v\,(u_x dx - u_y dy) + \iint_D vf\,dx\,dy \tag{5.8.13}$$

which is the solution of the Cauchy problem in terms of the Cauchy data given along the curve Λ. It is easy to see that the solution at the point (α, β) depends only on the Cauchy data along the arc QR on Λ. If the initial data were to change outside this arc QR, the solution would change only outside the triangle PQR. Thus, from Figure 5.8.2, we can see that each characteristic separates the region in which the solution remains unchanged from the region in which it varies. Because of this fact, the unique continuation of the solution across any characteristic is not possible. This is evident from Figure 5.8.2. The solution on the right of the characteristic P_1R_1 is determined by the initial data given in Q_1R_2, whereas the solution

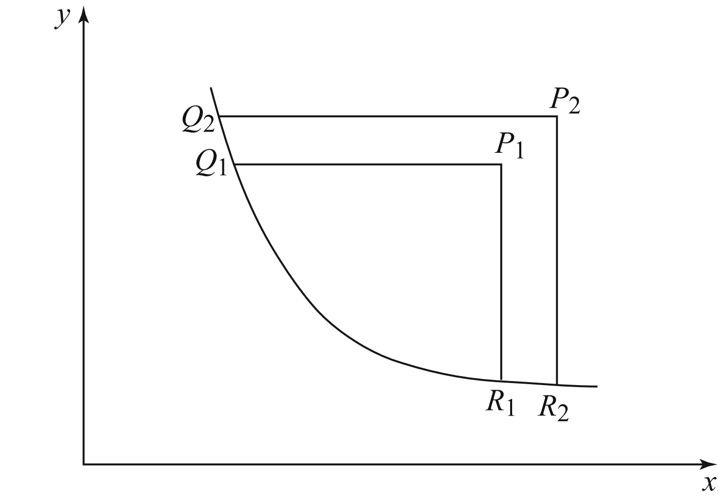

Figure 5.8.2 Solution on the right and left of the characteristic.

on the left is determined by the initial data given on Q_1R_1. If the initial data on R_1R_2 were changed, the solution on the right of P_1R_1 only will be affected.

It should be remarked here that the initial curve can intersect each characteristic at only one point. Suppose, for example, the initial curve Λ intersects the characteristic at two points, as shown in Figure 5.8.3. Then, the solution at P obtained from the initial data on QR will be different from the solution obtained from the initial data on RS. Hence, the Cauchy problem, in this case, is not solvable.

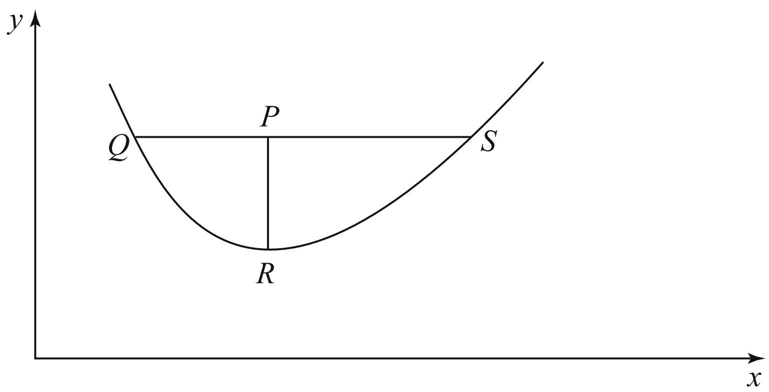

Figure 5.8.3 Initial curve intersects the characteristic at two points.

Example 5.8.1. The telegraph equation

$$w_{tt} + a^* w_t + b^* w = c^2 w_{xx},$$

solve it

may be transformed into canonical form

$$L[u] = u_{\xi\eta} + ku = 0,$$

by the successive transformations

$$w = u\, e^{-a^* t/2},$$

and

$$\xi = x + ct, \qquad \eta = x - ct,$$

where $k = \left(a^{*2} - 4b^*\right)/16c^2$.

We apply Riemann's method to determine the solution satisfying the initial conditions

$$u(x, 0) = f(x), \qquad u_t(x, 0) = g(x).$$

Since

$$t = \frac{1}{2c}(\xi - \eta),$$

the line $t = 0$ corresponds to the straight line $\xi = \eta$ in the $\xi - \eta$ plane. The initial conditions may thus be transformed into

$$[u]_{\xi=\eta} = f(\xi), \tag{5.8.14}$$

$$[u_\xi - u_\eta]_{\xi=\eta} = c^{-1} g(\xi). \tag{5.8.15}$$

We next determine the Riemann function $v(\xi, \eta; \alpha, \beta)$ which satisfies

$$v_{\xi\eta} + kv = 0, \tag{5.8.16}$$

$$v_\xi(\xi, \beta; \alpha, \beta) = 0, \tag{5.8.17}$$

$$v_\eta(\alpha, \eta; \alpha, \beta) = 0, \tag{5.8.18}$$

$$v(\alpha, \beta; \alpha, \beta) = 1. \tag{5.8.19}$$

The differential equation (5.8.16) is self-adjoint, that is,

$$L[v] = M[v] = v_{\xi\eta} + kv.$$

We assume that the Riemann function is of the form

$$v(\xi, \eta; \alpha, \beta) = F(s),$$

with the argument $s = (\xi - \alpha)(\eta - \beta)$. Substituting this value in equation (5.8.16), we obtain

$$sF_{ss} + F_s + kF = 0.$$

If we let $\lambda = \sqrt{4ks}$, the above equation becomes

$$F''(\lambda) + \frac{1}{\lambda} F'(\lambda) + F(\lambda) = 0.$$

This is the Bessel equation of order zero, and the solution is

$$F(\lambda) = J_0(\lambda),$$

disregarding $Y_0(\lambda)$ which is unbounded at $\lambda = 0$. Thus, the Riemann function is

$$v(\xi, \eta; \alpha, \beta) = J_0\left(\sqrt{4k(\xi - \alpha)(\eta - \beta)}\right)$$

which satisfies equation (5.8.16) and is equal to one on the characteristics $\xi = \alpha$ and $\eta = \beta$. Since $J_0'(0) = 0$, equations (5.8.17) and (5.8.18) are satisfied. From this, it immediately follows that

$$[v_\xi]_{\xi=\eta} = \frac{\sqrt{k}(\xi - \beta)}{\sqrt{(\xi - \alpha)(\eta - \beta)}} [J_0'(\lambda)]_{\xi=\eta},$$
$$[v_\eta]_{\xi=\eta} = \frac{\sqrt{k}(\xi - \alpha)}{\sqrt{(\xi - \alpha)(\eta - \beta)}} [J_0'(\lambda)]_{\xi=\eta}.$$

Thus, we have

$$[v_\xi - u_\eta]_{\xi=\eta} = \frac{\sqrt{k}(\alpha - \beta)}{\sqrt{(\xi - \alpha)(\xi - \beta)}} [J_0'(\lambda)]_{\xi=\eta}. \tag{5.8.20}$$

From the initial condition

$$u(Q) = f(\beta) \quad \text{and} \quad u(R) = f(\alpha), \tag{5.8.21}$$

and substituting equations (5.8.15), (5.8.19), and (5.8.20) into equation (5.8.13), we obtain

$$\begin{aligned} u(\alpha, \beta) = & \frac{1}{2}[f(\alpha) + f(\beta)] \\ & -\frac{1}{2}\int_\beta^\alpha \frac{\sqrt{k}(\alpha - \beta)}{\sqrt{(\tau - \alpha)(\tau - \beta)}} J_0'\left(\sqrt{4k(\tau - \alpha)(\tau - \beta)}\right) f(\tau)\, d\tau \\ & + \frac{1}{2c}\int_\beta^\alpha J_0\left(\sqrt{4k(\tau - \alpha)(\tau - \beta)}\right) g(\tau)\, d\tau. \end{aligned} \tag{5.8.22}$$

Replacing α and β by ξ and η, and substituting the original variables x and t, we obtain

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2}\int_{x-ct}^{x+ct} G(x,t,\tau)\, d\tau, \quad (5.8.23)$$

where

$$G(x,t,\tau) = \left\{-2\sqrt{k}\, ct f(\tau) J_0\left(\sqrt{4k\left[(\tau-x)^2 - c^2t^2\right]}\right)\right\} \Big/ \sqrt{(\tau-x)^2 - c^2t^2} + c^{-1} g(\tau) J_0\left(\sqrt{4k\left[(\tau-x)^2 - c^2t^2\right]}\right).$$

If we set $k = 0$, we arrive at the d'Alembert solution for the wave equation

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\, d\tau.$$

5.9 Solution of the Goursat Problem

The Goursat problem is that of finding the solution of a linear hyperbolic equation

$$u_{xy} = a_1(x,y)\, u_x + a_2(x,y)\, u_y + a_3(x,y)\, u + h(x,y), \quad (5.9.1)$$

satisfying the prescribed conditions

$$u(x,y) = f(x), \quad (5.9.2)$$

on a characteristic, say, $y = 0$, and

$$u(x,y) = g(x) \quad (5.9.3)$$

on a monotonic increasing curve $y = y(x)$ which, for simplicity, is assumed to intersect the characteristic at the origin.

The solution in the region between the x-axis and the monotonic curve in the first quadrant can be determined by the method of successive approximations. The proof is given in Garabedian (1964).

Example 5.9.1. Determine the solution of the Goursat problem

$$u_{tt} = c^2 u_{xx}, \quad (5.9.4)$$

$$u(x,t) = f(x), \quad \text{on} \quad x - ct = 0, \quad (5.9.5)$$

Do this

$$u(x,t) = g(x), \quad \text{on} \quad t = t(x), \quad (5.9.6)$$

where $f(0) = g(0)$.

The general solution of the wave equation is

$$u(x,t) = \phi(x+ct) + \psi(x-ct).$$

Applying the prescribed conditions, we obtain

$$f(x) = \phi(2x) + \psi(0), \tag{5.9.7}$$
$$g(x) = \phi(x + c\,t(x)) + \psi(x - c\,t(x)). \tag{5.9.8}$$

It is evident that

$$f(0) = \phi(0) + \psi(0) = g(0).$$

Now, if $s = x - c\,t(x)$, the inverse of it is $x = \alpha(s)$. Thus, equation (5.9.8) may be written as

$$g(\alpha(s)) = \phi(x + c\,t(x)) + \psi(s). \tag{5.9.9}$$

Replacing x by $(x + c\,t(x))/2$ in equation (5.9.7), we obtain

$$f\left(\frac{x + c\,t(x)}{2}\right) = \phi(x + c\,t(x)) + \psi(0). \tag{5.9.10}$$

Thus, using (5.9.10), equation (5.9.9) becomes

$$\psi(s) = g(\alpha(s)) - f\left(\frac{\alpha(s) + c\,t(\alpha(s))}{2}\right) + \psi(0).$$

Replacing s by $x - ct$, we have

$$\psi(x-ct) = g(\alpha(x-ct)) - f\left(\frac{\alpha(x-c\,t) + c\,t(\alpha(x-c\,t))}{2}\right) + \psi(0).$$

Hence, the solution is given by

$$u(x,t) = f\left(\frac{x+c\,t}{2}\right) - f\left(\frac{\alpha(x-c\,t) + c\,t(\alpha(x-c\,t))}{2}\right) + g(\alpha(x-c\,t)). \tag{5.9.11}$$

Let us consider a special case when the curve $t = t(x)$ is a straight line represented by $t - kx = 0$ with a constant $k > 0$. Then $s = x - ckx$ and hence $x = s/(1-ck)$. Using these values in (5.9.11), we obtain

$$u(x,t) = f\left(\frac{x+c\,t}{2}\right) - f\left(\frac{(1+c\,k)(x-c\,t)}{2(1-c\,k)}\right) + g\left(\frac{x-c\,t}{1-c\,k}\right). \tag{5.9.12}$$

When the values of u are prescribed on both characteristics, the problem of finding u of a linear hyperbolic equation is called a *characteristic initial-value problem.* This is a degenerate case of the Goursat problem.

Consider the characteristic initial-value problem

$$u_{xy} = h(x, y), \tag{5.9.13}$$
$$u(x, 0) = f(x), \tag{5.9.14}$$
$$u(0, y) = g(y), \tag{5.9.15}$$

where f and g are continuously differentiable, and $f(0) = g(0)$.

Integrating equation (5.9.13), we obtain

$$u(x, y) = \int_0^x \int_0^y h(\xi, \eta)\, d\eta\, d\xi + \phi(x) + \psi(y), \tag{5.9.16}$$

where ϕ and ψ are arbitrary functions. Applying the prescribed conditions (5.9.14) and (5.9.15), we have

$$u(x, 0) = \phi(x) + \psi(0) = f(x), \tag{5.9.17}$$
$$u(0, y) = \phi(0) + \psi(y) = g(y). \tag{5.9.18}$$

Thus,

$$\phi(x) + \psi(y) = f(x) + g(y) - \phi(0) - \psi(0). \tag{5.9.19}$$

But from (5.9.17), we have

$$\phi(0) + \psi(0) = f(0). \tag{5.9.20}$$

Hence, from (5.9.16), (5.9.19) and (5.9.20), we obtain

$$u(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y h(\xi, \eta)\, d\eta\, d\xi. \tag{5.9.21}$$

Example 5.9.2. Determine the solution of the characteristic initial-value problem

Do this

$$u_{tt} = c^2 u_{xx},$$
$$u(x, t) = f(x) \quad \text{on} \quad x + ct = 0,$$
$$u(x, t) = g(x) \quad \text{on} \quad x - ct = 0,$$

where $f(0) = g(0)$.

Here it is not necessary to reduce the given equation to canonical form. The general solution of the wave equation is

$$u(x, t) = \phi(x + ct) + \psi(x - ct).$$

The characteristics are

$$x + ct = 0, \qquad x - ct = 0.$$

Applying the prescribed conditions, we have

$$u(x,t) = \phi(2x) + \psi(0) = f(x) \quad \text{on} \quad x + ct = 0, \qquad (5.9.22)$$
$$u(x,t) = \phi(0) + \psi(2x) = g(x) \quad \text{on} \quad x - ct = 0. \qquad (5.9.23)$$

We observe that these equations are compatible, since $f(0) = g(0)$.

Now, replacing x by $(x+ct)/2$ in equation (5.9.22) and replacing x by $(x-ct)/2$ in equation (5.9.23), we have

$$\phi(x+ct) = f\left(\frac{x+ct}{2}\right) - \psi(0),$$
$$\phi(x-ct) = g\left(\frac{x-ct}{2}\right) - \phi(0).$$

Hence, the solution is given by

$$u(x,t) = f\left(\frac{x+ct}{2}\right) + g\left(\frac{x-ct}{2}\right) - f(0). \qquad (5.9.24)$$

We note that this solution can be obtained by substituting $k = -1/c$ into (5.9.12).

Example 5.9.3. Find the solution of the characteristic initial-value problem

$$y^3 u_{xx} - y u_{yy} + u_y = 0, \qquad (5.9.25)$$
$$u(x,y) = f(x) \quad \text{on} \quad x + \frac{y^2}{2} = 4 \quad \text{for} \quad 2 \le x \le 4,$$

Do this

$$u(x,y) = g(x) \quad \text{on} \quad x - \frac{y^2}{2} = 0 \quad \text{for} \quad 0 \le x \le 2,$$

with $f(2) = g(2)$.

Since the equation is hyperbolic except for $y = 0$, we reduce it to the canonical form

$$u_{\xi\eta} = 0,$$

where $\xi = x + (y^2/2)$ and $\eta = x - (y^2/2)$. Thus, the general solution is

$$u(x,y) = \phi\left(x + \frac{y^2}{2}\right) + \psi\left(x - \frac{y^2}{2}\right). \qquad (5.9.26)$$

Applying the prescribed conditions, we have

$$f(x) = \phi(4) + \psi(2x-4), \qquad (5.9.27)$$
$$g(x) = \phi(2x) + \psi(0). \qquad (5.9.28)$$

Now, if we replace $(2x-4)$ by $(x - y^2/2)$ in (5.9.27) and $(2x)$ by $(x + y^2/2)$ in (5.9.28), we obtain

$$\psi\left(x - \frac{y^2}{2}\right) = f\left(\frac{x}{2} - \frac{y^2}{4} + 2\right) - \phi(4),$$
$$\phi\left(x + \frac{y^2}{2}\right) = g\left(\frac{x}{2} + \frac{y^2}{4}\right) - \psi(0).$$

Thus,

$$u(x,y) = f\left(\frac{x}{2} - \frac{y^2}{4} + 2\right) + g\left(\frac{x}{2} + \frac{y^2}{4}\right) - \phi(4) - \psi(0).$$

But from (5.9.27) and (5.9.28), we see that

$$f(2) = \phi(4) + \psi(0) = g(2).$$

Hence,

$$u(x,y) = f\left(\frac{x}{2} - \frac{y^2}{4} + 2\right) + g\left(\frac{x}{2} + \frac{y^2}{4}\right) - f(2).$$

5.10 Spherical Wave Equation

convert from cartesian coordinates to spherical coordinates

In spherical polar coordinates (r, θ, ϕ), the wave equation (3.1.1) takes the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad (5.10.1)$$

Solutions of this equation are called *spherical symmetric waves* if u depends on r and t only. Thus, the solution $u = u(r,t)$ which satisfies the wave equation with spherical symmetry in three-dimensional space is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \quad (5.10.2)$$

Introducing a new dependent variable $U = ru(r,t)$, this equation reduces to a simple form

$$U_{tt} = c^2 U_{rr}. \quad (5.10.3)$$

This is identical with the one-dimensional wave equation (5.3.1) and has the general solution in the form

$$U(r,t) = \phi(r+ct) + \psi(r-ct), \quad (5.10.4)$$

or, equivalently,

$$u(r,t) = \frac{1}{r}\left[\phi(r+ct) + \psi(r-ct)\right]. \quad (5.10.5)$$

This solution consists of two progressive spherical waves traveling with constant velocity c. The terms involving ϕ and ψ represent the incoming waves to the origin and the outgoing waves from the origin respectively.

Physically, the solution for only outgoing waves generated by a source is of most interest, and has the form

$$u(r,t) = \frac{1}{r}\,\psi(r-ct), \tag{5.10.6}$$

where the explicit form of ψ is to be determined from the properties of the source. In the context of fluid flows, u represents the velocity potential so that the limiting total flux through a sphere of center at the origin and radius r is

$$Q(t) = \lim_{r\to 0} 4\pi r^2 u_r(r,t) = -4\pi\,\psi(-ct). \tag{5.10.7}$$

In physical terms, we say that there is a simple (or monopole) point source of strength $Q(t)$ located at the origin. Thus, the solution (5.10.6) can be expressed in terms of Q as

$$u(r,t) = -\frac{1}{4\pi r}\,Q\left(t-\frac{r}{c}\right). \tag{5.10.8}$$

This represents the velocity potential of the point source, and u_r is called the *radial velocity*. In fluid flows, the difference between the pressure at any time t and the equilibrium value is given by

$$p - p_0 = \rho\,u_t = -\frac{\rho}{4\pi r}\,\dot{Q}\left(t-\frac{r}{c}\right), \tag{5.10.9}$$

where ρ is the density of the fluid.

Following an analysis similar to Section 5.3, the solution of the initial-value problem with the initial data

$$u(r,0) = f(r), \quad u_t(r,0) = g(r), \quad r \geq 0, \tag{5.10.10}$$

where f and g are continuously differentiable, is given by

$$u(r,t) = \frac{1}{2r}\left[(r+ct)\,f(r+ct) + (r-ct)\,f(r-ct) + \frac{1}{c}\int_{r-ct}^{r+ct} \tau g(\tau)\,d\tau\right], \tag{5.10.11}$$

provided $r \geq ct$. However, when $r < ct$, this solution fails because f and g are not defined for $r < 0$. This initial data at $t = 0$, $r \geq 0$ determine the solution $u(r,t)$ only up to the characteristic $r = ct$ in the r-t plane. To find u for $r < ct$, we require u to be finite at $r = 0$ for all $t \geq 0$, that is, $U = 0$ at $r = 0$. Thus, the solution for $U(r,t)$ is

$$U(r,t) = \frac{1}{2}\left[(r+ct)\, f(r+ct) + (r-ct)\, f(r-ct) + \frac{1}{c}\int_{r-ct}^{r+ct} \tau g(\tau)\, d\tau\right], \tag{5.10.12}$$

provided $r \geq ct \geq 0$, and

$$U(r,t) = \frac{1}{2}\left[\phi(ct+r) + \psi(ct-r)\right], \qquad ct \geq r \geq 0, \tag{5.10.13}$$

where

$$\phi(ct) + \psi(ct) = 0, \quad \text{for} \quad ct \geq 0. \tag{5.10.14}$$

In view of the fact that $U_r + \frac{1}{c}U_t$ is constant on each characteristic $r + ct =$ constant, it turns out that

$$\phi'(ct+r) = (r+ct)\, f'(r+ct) + f(r+ct) + \frac{1}{c}(r+ct)\, g(r+ct),$$

or

$$\phi'(ct) = ct f'(ct) + f(ct) + t\, g(ct).$$

Integration gives

$$\phi(t) = t f(t) + \frac{1}{c}\int_0^t \tau g(\tau)\, d\tau + \phi(0),$$

so that

$$\psi(t) = -t f(t) - \frac{1}{c}\int_0^t \tau g(\tau)\, d\tau - \phi(0).$$

Substituting these values into (5.10.13) and using $U(r,t) = r u(r,t)$, we obtain, for $ct > r$,

$$u(r,t) = \frac{1}{2r}\left[(ct+r)\, f(ct+r) - (ct-r)\, f(ct-r) + \frac{1}{c}\int_{ct-r}^{ct+r} \tau g(\tau)\, d\tau\right]. \tag{5.10.15}$$

5.11 Cylindrical Wave Equation

In cylindrical polar coordinates (R, θ, z), the wave equation (3.1.1) assumes the form

$$u_{RR} + \frac{1}{R}u_R + \frac{1}{R^2}u_{\theta\theta} + u_{zz} = \frac{1}{c^2}u_{tt}. \tag{5.11.1}$$

If u depends only on R and t, this equation becomes

convert from cartesian coordinates to cylindrical coordinates

$$u_{RR} + \frac{1}{R}\,u_R = \frac{1}{c^2}\,u_{tt}. \tag{5.11.2}$$

Solutions of (5.11.2) are called *cylindrical waves.*

In general, it is not easy to find the solution of (5.11.1). However, we shall solve this equation by using the method of separation of variables in Chapter 7. Here we derive the solution for outgoing cylindrical waves from the spherical wave solution (5.10.8). We assume that sources of constant strength $Q\,(t)$ per unit length are distributed uniformly on the z-axis. The solution for the cylindrical waves produced by the line source is given by the total disturbance

$$u\,(R,t) = -\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{1}{r}\,Q\left(t-\frac{r}{c}\right)dz = -\frac{1}{2\pi}\int_{0}^{\infty}\frac{1}{r}\,Q\left(t-\frac{r}{c}\right)dz, \tag{5.11.3}$$

where R is the distance from the z-axis so that $R^2 = \left(r^2 - z^2\right)$.

Substitution of $z = R\sinh\xi$ and $r = R\cosh\xi$ in (5.11.3) gives

$$u\,(R,t) = -\frac{1}{2\pi}\int_{0}^{\infty} Q\left(t-\frac{R}{c}\cosh\xi\right)d\xi. \tag{5.11.4}$$

This is usually considered as the cylindrical wave function due to a source of strength $Q\,(t)$ at $R = 0$. It follows from (5.11.4) that

$$u_{tt} = -\frac{1}{2\pi}\int_{0}^{\infty} Q''\left(t-\frac{R}{c}\cosh\xi\right)d\xi, \tag{5.11.5}$$

$$u_R = \frac{1}{2\pi c}\int_{0}^{\infty}\cosh\xi\,Q'\left(t-\frac{R}{c}\cosh\xi\right)d\xi, \tag{5.11.6}$$

$$u_{RR} = -\frac{1}{2\pi c^2}\int_{0}^{\infty}\cosh^2\xi\,Q''\left(t-\frac{R}{c}\cosh\xi\right)d\xi, \tag{5.11.7}$$

which give

$$\begin{aligned} c^2\left(u_{RR}+\frac{1}{R}\,u_R\right) - u_{tt} &= \frac{1}{2\pi}\int_{0}^{\infty}\frac{d}{d\xi}\left[\frac{c}{R}\,Q'\left(t-\frac{R}{c}\cosh\xi\right)\sinh\xi\right]d\xi \\ &= \lim_{\xi\to\infty}\left[\frac{c}{2\pi R}\,Q'\left(t-\frac{R}{c}\cosh\xi\right)\sinh\xi\right] = 0, \end{aligned}$$

provided the differentiation under the sign of integration is justified and the above limit is zero. This means that $u\,(R,t)$ satisfies the cylindrical wave equation (5.11.2).

In order to find the asymptotic behavior of the solution as $R \to 0$, we substitute $\cosh\xi = \frac{c(t-\zeta)}{R}$ into (5.11.4) and (5.11.6) to obtain

$$u = -\frac{1}{2\pi}\int_{-\infty}^{t-R/c}\frac{Q\,(\zeta)\,d\zeta}{\left[(t-\zeta)^2 - \frac{R^2}{c^2}\right]^{\frac{1}{2}}}, \tag{5.11.8}$$

$$u_R = \frac{1}{2\pi}\int_{-\infty}^{t-R/c}\left(\frac{t-\zeta}{R}\right)\frac{Q'\,(\zeta)\,d\zeta}{\left[(t-\zeta)^2 - \frac{R^2}{c^2}\right]^{\frac{1}{2}}}, \tag{5.11.9}$$

which, in the limit $R \to 0$, give

$$u_R \sim \frac{1}{2\pi R} \int_{-\infty}^{t} Q'(\zeta)\, d\zeta = \frac{1}{2\pi R}\, Q(t)\,. \tag{5.11.10}$$

This leads to the result

$$\lim_{R\to 0} 2\pi R\, u_R = Q(t)\,, \tag{5.11.11}$$

or

$$u(R,t) \sim \frac{1}{2\pi}\, Q(t) \log R \quad \text{as} \quad R \to 0. \tag{5.11.12}$$

We next investigate the nature of the cylindrical wave solution near the waterfront ($R = ct$) and in the far field ($R \to \infty$). We assume $Q(t) = 0$ for $t < 0$ so that the lower limit of integration in (5.11.8) may be taken to be zero, and the solution is non-zero for $\tau = t - \frac{R}{c} > 0$, where τ is the time passed after the arrival of the wavefront. Consequently, (5.11.8) becomes

$$u(R,t) = -\frac{1}{2\pi} \int_0^{\tau} \frac{Q(\zeta)\, d\zeta}{\left[(t-\zeta)\left(t-\zeta+\frac{2R}{c}\right)\right]^{\frac{1}{2}}}. \tag{5.11.13}$$

Since $0 < \zeta < \tau$, $\frac{2R}{c} > \frac{R}{c} > \tau > \tau - \zeta > 0$, so that the second factor under the radical is approximately equal to $\frac{2R}{c}$ when $R \gg c\tau$, and hence,

$$\begin{aligned} u(R,t) &\sim -\frac{1}{2\pi}\left(\frac{c}{2R}\right)^{\frac{1}{2}} \int_0^{\tau} \frac{Q(\zeta)\, d\zeta}{(t-\zeta)^{\frac{1}{2}}} = -\left(\frac{c}{2R}\right)^{\frac{1}{2}} q(\tau) \\ &= -\left(\frac{c}{2R}\right)^{\frac{1}{2}} q\left(t - \frac{R}{c}\right), \qquad R \gg \frac{ct}{2}, \end{aligned} \tag{5.11.14}$$

where

$$q(\tau) = \frac{1}{2\pi} \int_0^{\tau} \frac{Q(\zeta)\, d\zeta}{\sqrt{\tau - \zeta}}. \tag{5.11.15}$$

Evidently, the amplitude involved in the solution (5.11.14) decays like $R^{-\frac{1}{2}}$ for large $R\,(R \to \infty)$.

Example 5.11.1. Determine the asymptotic form of the solution (5.11.4) for a harmonically oscillating source of frequency ω.

We take the source in the form $Q(t) = q_0 \exp\left[-i(\omega + i\varepsilon)t\right]$, where ε is positive and small so that $Q(t) \to 0$ as $t \to -\infty$. The small imaginary part ε of ω will make insignificant contributions to the solution at finite time as $\varepsilon \to 0$. Thus, the solution (5.11.4) becomes

$$\begin{aligned} u(R,t) &= -\left(\frac{q_0}{2\pi}\right) e^{-i\omega t} \int_0^{\infty} \exp\left(\frac{i\omega R}{c} \cosh\xi\right) d\xi \\ &= -\left(\frac{iq_0}{4}\right) e^{-i\omega t} H_0^{(1)}\left(\frac{\omega R}{c}\right), \end{aligned} \tag{5.11.16}$$

where $H_0^{(1)}(z)$ is the Hankel function given by

$$H_0^{(1)}(z) = \frac{2}{\pi i} \int_0^\infty \exp(iz \cosh \xi)\, d\xi. \tag{5.11.17}$$

In view of the asymptotic expansion of $H_0^{(1)}(z)$ in the form

$$H_0^{(1)}(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \exp\left[i\left(z - \frac{\pi}{4}\right)\right], \qquad z \to \infty, \tag{5.11.18}$$

the asymptotic solution for $u(R,t)$ in the limit $\left(\frac{\omega R}{c}\right) \to \infty$ is

$$u(R,t) \sim -\left(\frac{iq_0}{4}\right)\left(\frac{2c}{\pi \omega R}\right)^{\frac{1}{2}} \exp\left[-i\left(\omega t - \frac{\omega R}{c} - \frac{\pi}{4}\right)\right].$$

This represents the cylindrical wave propagating with constant velocity c. The amplitude of the wave decays like $R^{-\frac{1}{2}}$ as $R \to \infty$.

Example 5.11.2. For a supersonic flow ($M > 1$) past a solid body of revolution, the perturbation potential Φ satisfies the cylindrical wave equation

$$\Phi_{RR} + \frac{1}{R}\Phi_R = N^2 \Phi_{xx}, \qquad N^2 = M^2 - 1,$$

where R is the distance from the path of the moving body and x is the distance from the nose of the body.

It follows from problem 12 in 3.9 Exercises that Φ satisfies the equation

$$\Phi_{yy} + \Phi_{zz} = N^2\, \Phi_{xx}.$$

This represents a two-dimensional wave equation with $x \leftrightarrow t$ and $N^2 \leftrightarrow \frac{1}{c^2}$. For a body of revolution with $(y,z) \leftrightarrow (R,\theta)$, $\frac{\partial}{\partial \theta} \equiv 0$, the above equation reduces to the cylindrical wave equation

$$\Phi_{RR} + \frac{1}{R}\Phi_R = \frac{1}{c^2}\Phi_{tt}.$$

5.12 Exercises

1. Determine the solution of each of the following initial-value problems:

   (a) $u_{tt} - c^2 u_{xx} = 0, \quad u(x,0) = 0, \quad u_t(x,0) = 1.$

   (b) $u_{tt} - c^2 u_{xx} = 0, \quad u(x,0) = \sin x, \quad u_t(x,0) = x^2.$

   (c) $u_{tt} - c^2 u_{xx} = 0, \quad u(x,0) = x^3, \quad u_t(x,0) = x.$

(d) $u_{tt} - c^2 u_{xx} = 0, \quad u(x,0) = \cos x, \quad u_t(x,0) = e^{-1}.$

(e) $u_{tt} - c^2 u_{xx} = 0, \quad u(x,0) = \log\left(1 + x^2\right), \quad u_t(x,0) = 2.$

(f) $u_{tt} - c^2 u_{xx} = 0, \quad u(x,0) = x, \quad u_t(x,0) = \sin x.$

2. Determine the solution of each of the following initial-value problems:

(a) $u_{tt} - c^2 u_{xx} = x, \quad u(x,0) = 0, \quad u_t(x,0) = 3.$

(b) $u_{tt} - c^2 u_{xx} = x + ct, \quad u(x,0) = x, \quad u_t(x,0) = \sin x.$

(c) $u_{tt} - c^2 u_{xx} = e^x, \quad u(x,0) = 5, \quad u_t(x,0) = x^2.$

(d) $u_{tt} - c^2 u_{xx} = \sin x, \quad u(x,0) = \cos x, \quad u_t(x,0) = 1 + x.$

(e) $u_{tt} - c^2 u_{xx} = xe^t, \quad u(x,0) = \sin x, \quad u_t(x,0) = 0.$

(f) $u_{tt} - c^2 u_{xx} = 2, \quad u(x,0) = x^2, \quad u_t(x,0) = \cos x.$

3. A gas which is contained in a sphere of radius R is at rest initially, and the initial condensation is given by s_0 inside the sphere and zero outside the sphere. The condensation is related to the velocity potential by

$$s(t) = \left(1/c^2\right) u_t,$$

at all times, and the velocity potential satisfies the wave equation

$$u_{tt} = \nabla^2 u.$$

Determine the condensation $s(t)$ for all $t > 0$.

4. Solve the initial-value problem

$$\begin{aligned} u_{xx} + 2u_{xy} - 3u_{yy} &= 0, \\ u(x,0) = \sin x, \qquad & u_y(x,0) = x. \end{aligned}$$

5. Find the longitudinal oscillation of a rod subject to the initial conditions

$$\begin{aligned} u(x,0) &= \sin x, \\ u_t(x,0) &= x. \end{aligned}$$

6. By using the Riemann method, solve the following problems:

(a) $\sin^2 \mu\, \phi_{xx} - \cos^2 \mu\, \phi_{yy} - \left(\lambda^2 \sin^2 \mu\, \cos^2 \mu\right) \phi = 0,$

$$\begin{aligned} \phi(0,y) &= f_1(y), \qquad & \phi(x,0) &= g_1(x), \\ \phi_x(0,y) &= f_2(y), \qquad & \phi_y(x,0) &= g_2(x). \end{aligned}$$

(b) $x^2 u_{xx} - t^2 u_{tt} = 0,$

$$u(x, t_1) = f(x), \qquad u_t(x, t_2) = g(x).$$

7. Determine the solution of the initial boundary-value problem

$$\begin{aligned} u_{tt} &= 4\,u_{xx}, && 0 < x < \infty, \quad t > 0, \\ u(x, 0) &= x^4, && 0 \le x < \infty, \\ u_t(x, 0) &= 0, && 0 \le x < \infty, \\ u(0, t) &= 0, && \qquad\qquad\quad t \ge 0. \end{aligned}$$

8. Determine the solution of the initial boundary-value problem

$$\begin{aligned} u_{tt} &= 9\,u_{xx}, && 0 < x < \infty, \quad t > 0, \\ u(x, 0) &= 0, && 0 \le x < \infty, \\ u_t(x, 0) &= x^3, && 0 \le x < \infty, \\ u_x(0, t) &= 0, && \qquad\qquad\quad t \ge 0. \end{aligned}$$

9. Determine the solution of the initial boundary-value problem

$$\begin{aligned} u_{tt} &= 16\,u_{xx}, && 0 < x < \infty, \quad t > 0, \\ u(x, 0) &= \sin x, && 0 \le x < \infty, \\ u_t(x, 0) &= x^2, && 0 \le x < \infty, \\ u(0, t) &= 0, && \qquad\qquad\quad t \ge 0. \end{aligned}$$

10. In the initial boundary-value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, && 0 < x < l, \quad t > 0, \\ u(x, 0) &= f(x), && 0 \le x \le l, \\ u_t(x, 0) &= g(x), && 0 \le x \le l, \\ u(0, t) &= 0, && \qquad\qquad t \ge 0, \end{aligned}$$

if f and g are extended as odd functions, show that $u(x, t)$ is given by the solution (5.4.5) for $x > ct$ and solution (5.4.6) for $x < ct$.

11. In the initial boundary-value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, && 0 < x < l, \quad t > 0, \\ u(x, 0) &= f(x), && 0 \le x \le l, \\ u_t(x, 0) &= g(x), && 0 \le x \le l, \\ u_x(0, t) &= 0, && \qquad\qquad t \ge 0, \end{aligned}$$

if f and g are extended as even functions, show that $u(x, t)$ is given by solution (5.4.8) for $x > ct$, and solution (5.4.9) for $x < ct$.

12. Determine the solution of the initial boundary-value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, && 0 < x < \infty, \quad t > 0, \\ u(x,0) &= f(x), && 0 \le x < \infty, \\ u_t(x,0) &= 0, && 0 \le x < \infty, \\ u_x(0,t) + h\,u(0,t) &= 0, && t \ge 0, \quad h = \text{constant}. \end{aligned}$$

State the compatibility condition of f.

13. Find the solution of the problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, && at < x < \infty, \quad t > 0, \\ u(x,0) &= f(x), && 0 < x < \infty, \\ u_t(x,0) &= 0, && 0 < x < \infty, \\ u(at,t) &= 0, && t > 0, \end{aligned}$$

where $f(0) = 0$ and a is constant.

14. Find the solution of the initial boundary-value problem

$$\begin{aligned} u_{tt} &= u_{xx}, && 0 < x < 2, \quad t > 0, \\ u(x,0) &= \sin(\pi x/2), && 0 \le x \le 2, \\ u_t(x,0) &= 0, && 0 \le x \le 2, \\ u(0,t) &= 0, \; u(2,t) = 0, && t \ge 0. \end{aligned}$$

15. Find the solution of the initial boundary-value problem

$$\begin{aligned} u_{tt} &= 4\,u_{xx}, && 0 < x < 1, \quad t > 0, \\ u(x,0) &= 0, && 0 \le x \le 1, \\ u_t(x,0) &= x(1-x), && 0 \le x \le 1, \\ u(0,t) &= 0, \; u(1,t) = 0, && t \ge 0. \end{aligned}$$

16. Determine the solution of the initial boundary-value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, && 0 < x < l, \quad t > 0, \\ u(x,0) &= f(x), && 0 \le x \le l, \\ u_t(x,0) &= g(x), && 0 \le x \le l, \\ u_x(0,t) &= 0, \; u_x(l,t) = 0, \; t \ge 0, \end{aligned}$$

by extending f and g as even functions about $x = 0$ and $x = l$.

17. Determine the solution of the initial boundary-value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, && 0 < x < l, \quad t > 0, \\ u(x,0) &= f(x), && 0 \le x \le l, \\ u_t(x,0) &= g(x), && 0 \le x \le l, \\ u(0,t) &= p(t), \; u(l,t) = q(t), \; t \ge 0. \end{aligned}$$

18. Determine the solution of the initial boundary-value problem

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx}, && 0 < x < l, \quad t > 0,\\
u(x,0) &= f(x), && 0 \le x \le l,\\
u_t(x,0) &= g(x), && 0 \le x \le l,\\
u_x(0,t) &= p(t),\ u_x(l,t) = q(t),\ t \ge 0.
\end{aligned}
$$

19. Solve the characteristic initial-value problem

$$
\begin{aligned}
xy^3 u_{xx} - x^3 y\, u_{yy} - y^3 u_x + x^3 u_y &= 0,\\
u(x,y) = f(x) \quad \text{on} \quad y^2 - x^2 = 8 \quad &\text{for} \quad 0 \le x \le 2,\\
u(x,y) = g(x) \quad \text{on} \quad y^2 + x^2 = 16 \quad &\text{for} \quad 2 \le x \le 4,
\end{aligned}
$$

with $f(2) = g(2)$.

20. Solve the Goursat problem

$$
\begin{aligned}
xy^3 u_{xx} - x^3 y\, u_{yy} - y^3 u_x + x^3 u_y &= 0,\\
u(x,y) = f(x) \quad \text{on} \quad y^2 + x^2 = 16 \quad &\text{for} \quad 0 \le x \le 4,\\
u(x,y) = g(y) \quad \text{on} \quad x = 0 \quad &\text{for} \quad 0 \le y \le 4,
\end{aligned}
$$

where $f(0) = g(4)$.

21. Solve

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx},\\
u(x,t) &= f(x) \quad \text{on} \quad t = t(x),\\
u(x,t) &= g(x) \quad \text{on} \quad x + ct = 0,
\end{aligned}
$$

where $f(0) = g(0)$.

22. Solve the characteristic initial-value problem

$$
\begin{aligned}
xu_{xx} - x^3 u_{yy} - u_x = 0, \quad x &\ne 0,\\
u(x,y) = f(y) \quad \text{on} \quad y - \frac{x^2}{2} = 0 \quad &\text{for} \quad 0 \le y \le 2,\\
u(x,y) = g(y) \quad \text{on} \quad y + \frac{x^2}{2} = 4 \quad &\text{for} \quad 2 \le y \le 4,
\end{aligned}
$$

where $f(2) = g(2)$.

23. Solve

$$
\begin{aligned}
u_{xx} + 10\, u_{xy} + 9\, u_{yy} &= 0,\\
u(x,0) &= f(x),\\
u_y(x,0) &= g(x).
\end{aligned}
$$

24. Solve

$$\begin{aligned} 4\,u_{xx} + 5\,u_{xy} + u_{yy} + u_x + u_y &= 2, \\ u(x,0) &= f(x), \\ u_y(x,0) &= g(x). \end{aligned}$$

25. Solve

$$\begin{aligned} 3\,u_{xx} + 10\,u_{xy} + 3\,u_{yy} &= 0, \\ u(x,0) = f(x), \quad u_y(x,0) &= g(x). \end{aligned}$$

26. Solve

$$\begin{aligned} u_{xx} - 3\,u_{xy} + 2\,u_{yy} &= 0, \\ u(x,0) = f(x), \quad u_y(x,0) &= g(x). \end{aligned}$$

27. Solve

$$\begin{aligned} x^2 u_{xx} - t^2 u_{tt} &= 0 \quad x > 0, \quad t > 0, \\ u(x,1) &= f(x), \\ u_t(x,1) &= g(x). \end{aligned}$$

28. Consider the initial boundary-value problem for a string of length l under the action of an external force $q(x,t)$ per unit length. The displacement $u(x,t)$ satisfies the wave equation

$$\rho\, u_{tt} = T u_{xx} + \rho\, q(x,t),$$

where ρ is the line density of the string and T is the constant tension of the string. The initial and boundary conditions of the problem are

$$\begin{aligned} &u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad 0 \le x \le l, \\ &u(0,t) = u(l,t) = 0, \qquad t > 0. \end{aligned}$$

Show that the energy equation is

$$\frac{dE}{dt} = \left[T u_x u_t\right]_0^l + \int_0^l \rho\, q\, u_t\, dx,$$

where E represents the energy integral

$$E(t) = \frac{1}{2}\int_0^l \left(\rho\, u_t^2 + T u_x^2\right) dx.$$

Explain the physical significance of the energy equation.

Hence or otherwise, derive the principle of conservation of energy, that is, that the total energy is constant for all $t \ge 0$ provided that the string has free or fixed ends and there are no external forces.

29. Show that the solution of the signaling problem governed by the wave equation

$$u_{tt} = c^2 u_{xx}, \qquad x > 0, \qquad t > 0,$$
$$u(x,0) = u_t(x,0) = 0, \qquad x > 0,$$
$$u(0,t) = U(t), \qquad t > 0,$$

is

$$u(x,t) = U\left(t - \frac{x}{c}\right) H\left(t - \frac{x}{c}\right),$$

where H is the Heaviside unit step function.

30. Obtain the solution of the initial-value problem of the homogeneous wave equation

$$u_{tt} - c^2 u_{xx} = \sin(kx - \omega t), \qquad -\infty < x < \infty, \qquad t > 0,$$
$$u(x,0) = 0 = u_t(x,0), \quad \text{for all} \quad x \in \mathbb{R},$$

where c, k and ω are constants.

Discuss the non-resonance case, $\omega \neq ck$ and the resonance case, $\omega = ck$.

31. In each of the following Cauchy problems, obtain the solution of the system

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x) \quad \text{for} \quad x \in \mathbb{R},$$

for the given c, $f(x)$ and $g(x)$:

(a) $c = 3$, $f(x) = \cos x$, $g(x) = \sin 2x$.

(b) $c = 1$, $f(x) = \sin 3x$, $g(x) = \cos 3x$.

(c) $c = 7$, $f(x) = \cos 3x$, $g(x) = x$.

(d) $c = 2$, $f(x) = \cosh x$, $g(x) = 2x$.

(e) $c = 3$, $f(x) = x^3$, $g(x) = x \cos x$.

(f) $c = 4$, $f(x) = \cos x$, $g(x) = xe^{-x}$.

32. If $u(x,t)$ is the solution of the nonhomogeneous Cauchy problem

$$u_{tt} - c^2 u_{xx} = p(x,t), \quad \text{for} \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(x,0) = 0 = u_t(x,0), \quad \text{for} \quad x \in \mathbb{R},$$

and if $v(x,t,\tau)$ is the solution of the nonhomogeneous Cauchy problem

$$\begin{aligned} v_{tt} - c^2 v_{xx} &= 0, \quad \text{for} \quad x \in \mathbb{R}, \quad t > 0, \\ v(x,0;\tau) &= 0, \quad v_t(x,0;\tau) = p(x,\tau), \quad x \in \mathbb{R}, \end{aligned}$$

show that

$$u(x,t) = \int_0^t v(x,t;\tau)\,d\tau.$$

This is known as the *Duhamel principle* for the wave equation.

33. Show that the solution of the nonhomogeneous diffusion equation with homogeneous boundary and initial data

$$\begin{aligned} u_t &= \kappa u_{xx} + p(x,t), \qquad 0 < x < l, \quad t > 0, \\ u(0,t) &= 0 = u(l,t), \qquad t > 0, \\ u(x,0) &= 0, \qquad 0 < x < l, \end{aligned}$$

is

$$u(x,t) = \int_0^t v(x,t;\tau)\,d\tau,$$

where $v = v(x,t;\tau)$ satisfies the homogeneous diffusion equation with nonhomogeneous boundary and initial data

$$\begin{aligned} v_{tt} &= \kappa v_{xx} + p(x,t), \qquad 0 < x < l, \quad t > 0, \\ v(0,t;\tau) &= 0 = v(l,t;\tau), \qquad t > 0, \\ v(x,\tau;\tau) &= p(x,\tau). \end{aligned}$$

This is known as the *Duhamel principle* for the diffusion equation.

34. Use the Duhamel principle to solve the nonhomogeneous diffusion equation

$$u_t = \kappa u_{xx} + e^{-t}\sin \pi x, \qquad 0 < x < l, \quad t > 0,$$

with the homogeneous boundary and initial data

$$\begin{aligned} u(0,t) &= 0, \quad u(1,t) = 0, \qquad t > 0, \\ u(x,0) &= 0, \qquad 0 \le x \le 1. \end{aligned}$$

35. (a) Verify that

$$u_n(x,y) = \exp\left(ny - \sqrt{n}\right)\sin nx,$$

is the solution of the Laplace equation

$$u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R}, \qquad y > 0,$$
$$u(x, 0) = 0, \qquad u_y(x, 0) = n \exp\left(-\sqrt{n}\right) \sin nx,$$

where n is a positive integer.
(b) Show that this Cauchy problem is not well posed.

36. Show that the following Cauchy problems are not well posed:

(a)
$$u_t = u_{xx}, \quad x \in \mathbb{R}, \qquad t > 0,$$
$$u(0, t) = \left(\tfrac{2}{n}\right) \sin\left(2n^2 t\right), \qquad u_x(0, t) = 0, \qquad t > 0.$$

(b)
$$u_{xx} + u_{yy} = 0, \qquad x \in \mathbb{R}, \qquad t > 0,$$
$$u_n(x, 0) \to 0, \quad (u_n)_y(x, 0) \to 0, \quad \text{as} \quad n \to \infty.$$